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Valuations and hyperplanes of dual polar spaces

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Abstract

Valuations were introduced in De Bruyn and Vandecasteele (Valuations of near polygons, preprint, 2004) as a very important tool for classifying near polygons. In the present paper we study valuations of dual polar spaces. We will introduce the class of the SDPS-valuations and characterize these valuations. We will show that a valuation of a finite thick dual polar space is the extension of an SDPS-valuation if and only if no induced hex valuation is ovoidal or semi-classical. Each SDPS-valuation will also give rise to a geometric hyperplane of the dual polar space.

MSC: 51A50; 51E12; 51E20

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1. Introduction

A near polygon [14] is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I}),\ \mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$, with the property that for every point p and every line L there exists a unique point on L nearest to p. Here distances $\mathrm{d}(\cdot, \cdot)$ are measured in the point graph or collinearity graph $\Gamma_{\mathcal{S}}$ of $\mathcal{S}$. If d denotes the diameter of $\Gamma_{\mathcal{S}}$, then the near polygon is called a *near* $2d$*-gon*. A near 0-gon is a point and a near 2-gon is a line.

If X_1 and X_2 are two nonempty set of points of a near polygon, then $\mathrm{d}(X_1, X_2)$ denotes the minimal distance between a point of X_1 and a point of X_2. If X_1 is a singleton $\{x_1\}$, then we will also write $\mathrm{d}(x_1, X_2)$ instead of $\mathrm{d}(\{x_1\}, X_2)$. For every $i \in \mathbb{N}$ and for every

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nonempty set X of points, $\Gamma_i(X)$ denotes the set of all points y for which $\mathrm{d}(y, X) = i$. If X is a singleton $\{x\}$, then we will also write $\Gamma_i(x)$ instead of $\Gamma_i(\{x\})$.

A near $2d$-gon, $d \geqslant 2$, is called a *generalized* $2d$*-gon* [16] if $|\Gamma_{i-1}(x) \cap \Gamma_1(y)| = 1$ for every $i \in \{1, \ldots, d-1\}$ and every two points x and y at distance i from each other. The generalized quadrangles [9] are precisely the near quadrangles. A generalized $2d$-gon is called *degenerate* if it does not contain an ordinary $2d$-gon as subgeometry. A degenerate generalized quadrangle consists of a number of lines through a point.

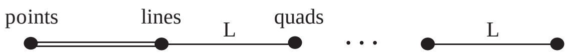
A subspace X of a near polygon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is called *geodetically closed* if every point on a shortest path between two points of X also belongs to X. Suppose X is geodetically closed. Let $\mathcal{L}_X$ denote the set of lines of $\mathcal{S}$ which are completely contained in X and put $\mathrm{I}_X := \mathrm{I} \cap (X \times \mathcal{L}_X)$. Then $(X, \mathcal{L}_X, \mathrm{I}_X)$ is a (geodetically closed) sub near polygon of $\mathcal{S}$. A nondegenerate geodetically closed sub near quadrangle of a near polygon is called a *quad*. Sufficient conditions for the existence of quads were given in Proposition 2.5 of [14].

If $X_1, \ldots, X_k$ are nonempty sets of points, then $\mathcal{C}(X_1, \ldots, X_k)$ denotes the smallest geodetically closed sub near polygon through $X_1 \cup \cdots \cup X_k$, i.e. the intersection of all geodetically closed sub near polygons through $X_1 \cup \cdots \cup X_k$. If one of these sets is a singleton $\{x\}$, then we will often omit the braces and write $\mathcal{C}(\cdots, x, \cdots)$ instead of $\mathcal{C}(\cdots, \{x\}, \ldots)$.

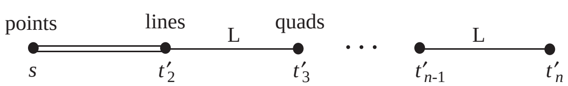
A geodetically closed sub near polygon F of a dense near polygon $\mathcal{S}$ is called *classical in* $\mathcal{S}$ if for every point x of $\mathcal{S}$, there exists a (necessarily unique) point $\pi_F(x)$ in F such that $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_F(x)) + \mathrm{d}(\pi_F(x), y)$ for every point y of F. We call π_F the *projection* on F.

A near polygon is said to have *order* (s, t) if every line is incident with $s+1$ points and if every point is incident with precisely $t+1$ lines. A near $2n$-gon, $n \geqslant 2$, is called *regular* if it has an order (s, t) and if there exist constants t_i, $i \in \{0, \ldots, n\}$, such that for every two points x and y at distance i from each other, there are precisely $t_i + 1$ lines through y containing a (necessarily unique) point at distance $i-1$ from x. Obviously, $t_0 = -1$, $t_1 = 0$ and $t_n = t$.

A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbours. If x and y are two points of a dense near polygon at distance δ from each other, then by Theorem 4 of [2], x and y are contained in a unique geodetically closed sub near 2δ-gon (which necessarily coincides with $\mathcal{C}(x, y)$). These sub near polygons are called *hexes* if δ is equal to 3. For δ equal to 0, 1, respectively 2, we find the points, lines, respectively quads, of $\mathcal{S}$. With every dense near $2n$-gon $\mathcal{S}$, there is associated a rank n geometry Δ. The elements of type $i \in \{1, \ldots, n\}$ are the geodetically closed sub near $2(i-1)$-gons of $\mathcal{S}$. The geometry Δ has the following diagram:



If $\mathcal{S}$ is a regular near $2n$-gon with parameters s, t, t_i $(0 \leqslant i \leqslant n)$ such that $s \geqslant 2$ and $t_2 \geqslant 1$, then we can parametrize the diagram as follows:



Here $t'_i := \frac{t_i - t_{i-1}}{t_{i-1} - t_{i-2}}$ for every $i \in \{2, \ldots, n\}$. Note that $t'_2 = t_2$.