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Note

On the asymptotic number of non-equivalent q-ary linear codes

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Abstract

Let $\mathfrak{M}_{n,q} \subset \mathrm{GL}(n, \mathbb{F}_q)$ be the group of monomial matrices, i.e., the group generated by all permutation matrices and diagonal matrices in $\mathrm{GL}(n, \mathbb{F}_q)$. The group $\mathfrak{M}_{n,q}$ acts on the set $\mathcal{V}(\mathbb{F}_q^n)$ of all subspaces of $\mathbb{F}_q^n$. The number of orbits of this action, denoted by $N_{n,q}$, is the number of non-equivalent linear codes in $\mathbb{F}_q^n$. It was conjectured by Lax that $N_{n,q} \sim \frac{|\mathcal{V}(\mathbb{F}_q^n)|}{n!(q-1)^{n-1}}$ as $n \to \infty$. We confirm this conjecture in this paper.

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1. Introduction

Let $\mathbb{F}_q$ be the finite field with q elements. Let $\mathfrak{S}_n$ be the subgroup of all permutation matrices in $\mathrm{GL}(n, \mathbb{F}_q)$ and $\mathfrak{D}_{n,q}$ the subgroup of all diagonal matrices in $\mathrm{GL}(n, \mathbb{F}_q)$. The subgroup of $\mathrm{GL}(n, \mathbb{F}_q)$ generated by $\mathfrak{S}_n \cup \mathfrak{D}_{n,q}$ is the group of monomial matrices and is denoted by $\mathfrak{M}_{n,q}$. $\mathfrak{M}_{n,q}$ is the image of a faithful representation of the wreath product $\mathbb{F}_q^\times$ wr S_n, where $\mathbb{F}_q^\times$ is the multiplicative group of $\mathbb{F}_q$ and S_n is the symmetric group on $\{1, \ldots, n\}$.

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The action of $\mathfrak{M}_{n,q}$ on $\mathbb{F}_q^n$ is inherited from that of $\mathrm{GL}(n, \mathbb{F}_q)$. Let $\mathcal{V}(\mathbb{F}_q^n)$ be the set of all subspaces of $\mathbb{F}_q^n$. Then $\mathfrak{M}_{n,q}$ acts on $\mathcal{V}(\mathbb{F}_q^n)$ in the natural way. The $\mathfrak{M}_{n,q}$-orbits in $\mathcal{V}(\mathbb{F}_q^n)$ are the equivalence classes of q-ary linear codes of length n.

Let $N_{n,q}$ be the number of $\mathfrak{M}_{n,q}$-orbits in $\mathcal{V}(\mathbb{F}_q^n)$. For general n, no explicit formula for $N_{n,q}$ is known. For specific n and q, not too large, $N_{n,q}$ can be computed using the Burnside lemma, but the computations are complicated and rely on computer assistance. The problem considered in this paper is the asymptotic behavior of $N_{n,q}$ as $n \to \infty$.

For two sequences of real numbers $f(n)$ and $g(n)$, $f(n) \sim g(n)$ means that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$; $f = O\big(g(n)\big)$ means that there exists a constant $A > 0$, such that

$$|f(n)| \leqslant A|g(n)| \quad \text{for } n \text{ sufficiently large.}$$

Observe that the action of $\mathfrak{M}_{n,q}$ on $\mathcal{V}(\mathbb{F}_q^n)$ has a kernel

$$\Delta := \{aI : a \in \mathbb{F}_q^\times\},$$

where $I \in \mathrm{GL}(n, \mathbb{F}_q)$ is the identity matrix. Recently, Lax [7] conjectured that

$$N_{n,q} \sim \frac{|\mathcal{V}(\mathbb{F}_q^n)|}{|\mathfrak{M}_{n,q}/\Delta|} \quad \text{as } n \to \infty. \tag{1.1}$$

Put

$$G_{n,q} = |\mathcal{V}(\mathbb{F}_q^n)| = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q,$$

where $[{n \atop i}]_q$ is the q-binomial coefficient. Then (1.1) can be written as

$$N_{n,q} \sim \frac{G_{n,q}}{n!(q-1)^{n-1}}. \tag{1.2}$$

For each $n \times n$ matrix T over $\mathbb{F}_q$, let $\mathcal{L}(T)$ be the set of all T-invariant subspaces of $\mathbb{F}_q^n$. From the Burnside lemma, we have

$$\begin{aligned} N_{n,q} &= \frac{1}{n!(q-1)^n} \sum_{T \in \mathfrak{M}_{n,q}} |\mathcal{L}(T)| \\ &= \frac{G_{n,q}}{n!(q-1)^{n-1}} + \frac{1}{n!(q-1)^n} \sum_{T \in \mathfrak{M}_{n,q} \setminus \Delta} |\mathcal{L}(T)|. \end{aligned}$$

Therefore, statement (1.2) is equivalent to

$$\frac{1}{G_{n,q}} \sum_{T \in \mathfrak{M}_{n,q} \setminus \Delta} |\mathcal{L}(T)| \to 0 \quad \text{as } n \to \infty. \tag{1.3}$$

It is known that $\frac{1}{G_{n,q}} = O(q^{-\frac{1}{4}n^2})$ [9]. On the other hand, denoting the conjugacy class of T in $\mathfrak{M}_{n,q}$ by $[T]$ and the total number of conjugacy classes of $\mathfrak{M}_{n,q}$ by $c(n, q)$, we obviously have

$$\sum_{T \in \mathfrak{M}_{n,q} \setminus \Delta} |\mathcal{L}(T)| < c(n, q) \max_{T \in \mathfrak{M}_{n,q} \setminus \Delta} |[T]|\, |\mathcal{L}(T)|. \tag{1.4}$$