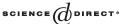


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Note

On the asymptotic number of non-equivalent q-ary linear codes

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Abstract

Let $\mathfrak{M}_{n,q} \subset \mathrm{GL}(n,\mathbb{F}_q)$ be the group of monomial matrices, i.e., the group generated by all permutation matrices and diagonal matrices in $GL(n, \mathbb{F}_q)$. The group $\mathfrak{M}_{n,q}$ acts on the set $\mathcal{V}(\mathbb{F}_q^n)$ of all subspaces of \mathbb{F}_q^n . The number of orbits of this action, denoted by $N_{n,q}$, is the number of nonequivalent linear codes in \mathbb{F}_q^n . It was conjectured by Lax that $N_{n,q} \sim \frac{|\mathcal{V}(\mathbb{F}_q^n)|}{n!(q-1)^{n-1}}$ as $n \to \infty$. We confirm this conjecture in this paper. © 2005 Elsevier Inc. All rights reserved.

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1. Introduction

Let \mathbb{F}_q be the finite field with q elements. Let \mathfrak{S}_n be the subgroup of all permutation matrices in $GL(n, \mathbb{F}_q)$ and $\mathfrak{D}_{n,q}$ the subgroup of all diagonal matrices in $GL(n, \mathbb{F}_q)$. The subgroup of $\mathrm{GL}(n,\mathbb{F}_q)$ generated by $\mathfrak{S}_n \cup \mathfrak{D}_{n,q}$ is the group of monomial matrices and is denoted by $\mathfrak{M}_{n,q}$. $\mathfrak{M}_{n,q}$ is the image of a faithful representation of the wreath product \mathbb{F}_q^{\times} wr S_n , where \mathbb{F}_q^{\times} is the multiplicative group of \mathbb{F}_q and S_n is the symmetric group on $\{1, \ldots, n\}.$

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The action of $\mathfrak{M}_{n,q}$ on \mathbb{F}_q^n is inherited from that of $\mathrm{GL}(n,\mathbb{F}_q)$. Let $\mathcal{V}(\mathbb{F}_q^n)$ be the set of all subspaces of \mathbb{F}_q^n . Then $\mathfrak{M}_{n,q}$ acts on $\mathcal{V}(\mathbb{F}_q^n)$ in the natural way. The $\mathfrak{M}_{n,q}$ -orbits in $\mathcal{V}(\mathbb{F}_q^n)$ are the equivalence classes of q-ary linear codes of length n.

Let $N_{n,q}$ be the number of $\mathfrak{M}_{n,q}$ -orbits in $\mathcal{V}(\mathbb{F}_q^n)$. For general n, no explicit formula for $N_{n,q}$ is known. For specific n and q, not too large, $N_{n,q}$ can be computed using the Burnside lemma, but the computations are complicated and rely on computer assistance. The problem considered in this paper is the asymptotic behavior of $N_{n,q}$ as $n \to \infty$.

For two sequences of real numbers f(n) and g(n), $f(n) \sim g(n)$ means that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$; f = O(g(n)) means that there exists a constant A > 0, such that

$$|f(n)| \le A|g(n)|$$
 for *n* sufficiently large.

Observe that the action of $\mathfrak{M}_{n,q}$ on $\mathcal{V}(\mathbb{F}_q^n)$ has a kernel

$$\Delta:=\{aI:a\in\mathbb{F}_q^\times\},$$

where $I \in GL(n, \mathbb{F}_q)$ is the identity matrix. Recently, Lax [7] conjectured that

$$N_{n,q} \sim \frac{|\mathcal{V}(\mathbb{F}_q^n)|}{|\mathfrak{M}_{n,q}/\Delta|} \quad \text{as } n \to \infty.$$
 (1.1)

Put

$$G_{n,q} = |\mathcal{V}(\mathbb{F}_q^n)| = \sum_{i=0}^n \begin{bmatrix} n \\ i \end{bmatrix}_q,$$

where $\begin{bmatrix} n \\ i \end{bmatrix}_q$ is the *q*-binomial coefficient. Then (1.1) can be written as

$$N_{n,q} \sim \frac{G_{n,q}}{n!(q-1)^{n-1}}.$$
 (1.2)

For each $n \times n$ matrix T over \mathbb{F}_q , let $\mathcal{L}(T)$ be the set of all T-invariant subspaces of \mathbb{F}_q^n . From the Burnside lemma, we have

$$N_{n,q} = \frac{1}{n!(q-1)^n} \sum_{T \in \mathfrak{M}_{n,q}} |\mathcal{L}(T)|$$

$$= \frac{G_{n,q}}{n!(q-1)^{n-1}} + \frac{1}{n!(q-1)^n} \sum_{T \in \mathfrak{M}_{n,q} \setminus \Lambda} |\mathcal{L}(T)|.$$

Therefore, statement (1.2) is equivalent to

$$\frac{1}{G_{n,q}} \sum_{T \in \mathfrak{M}_{n,q} \setminus \Delta} |\mathcal{L}(T)| \to 0 \quad \text{as } n \to \infty.$$
 (1.3)

It is known that $\frac{1}{G_{n,q}} = O(q^{-\frac{1}{4}n^2})$ [9]. On the other hand, denoting the conjugacy class of T in $\mathfrak{M}_{n,q}$ by [T] and the total number of conjugacy classes of $\mathfrak{M}_{n,q}$ by c(n,q), we obviously have

$$\sum_{T \in \mathfrak{M}_{n,q} \setminus \Delta} |\mathcal{L}(T)| < c(n,q) \max_{T \in \mathfrak{M}_{n,q} \setminus \Delta} |[T]| |\mathcal{L}(T)|. \tag{1.4}$$

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