Available online at www.sciencedirect.com
SCIENCE DIRECT®

Journal of Combinatorial Theory, Series A 112 (2005) 44–81

Journal of Combinatorial Theory Series A

www.elsevier.com/locate/jcta

Tableaux on $k+1$-cores, reduced words for affine permutations, and k-Schur expansions

Luc Lapointe[a,1], Jennifer Morse[b,2]

[a] *Instituto de Matemática y Física, Universidad de Talca, Casilla 747, Talca, Chile*
[b] *Department of Mathematics, University of Miami, Coral Gables, FL 33124, USA*

Received 11 August 2003

Communicated by Adriano Garsia
Available online 17 March 2005

Abstract

The k-Young lattice Y^k is a partial order on partitions with no part larger than k. This weak subposet of the Young lattice originated (Duke Math. J. 116 (2003) 103–146) from the study of the k-Schur functions $s_\lambda^{(k)}$, symmetric functions that form a natural basis of the space spanned by homogeneous functions indexed by k-bounded partitions. The chains in the k-Young lattice are induced by a Pieri-type rule experimentally satisfied by the k-Schur functions. Here, using a natural bijection between k-bounded partitions and $k+1$-cores, we establish an algorithm for identifying chains in the k-Young lattice with certain tableaux on $k+1$ cores. This algorithm reveals that the k-Young lattice is isomorphic to the weak order on the quotient of the affine symmetric group $\tilde{S}_{k+1}$ by a maximal parabolic subgroup. From this, the conjectured k-Pieri rule implies that the k-Kostka matrix connecting the homogeneous basis $\{h_\lambda\}_{\lambda\in Y^k}$ to $\{s_\lambda^{(k)}\}_{\lambda\in Y^k}$ may now be obtained by counting appropriate classes of tableaux on $k+1$-cores. This suggests that the conjecturally positive k-Schur expansion coefficients for Macdonald polynomials (reducing to q,t-Kostka polynomials for large k) could be described by a q,t-statistic on these tableaux, or equivalently on reduced words for affine permutations.

Keywords: Affine Weyl group; Cores; k-Schur functions; Macdonald polynomials

E-mail addresses: lapointe@inst-mat.utalca.cl (L. Lapointe), morsej@math.miami.edu (J. Morse).

[1] Research of L. Lapointe was supported in part by FONDECYT (Chile) Grant #1030114, the Programa Formas Cuadráticas of the Universidad de Talca, and NSERC (Canada) Grant #250904.

[2] Research of J. Morse was supported in part by NSF Grant #DMS-0400628.

0097-3165/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.jcta.2005.01.003

Contents

1. Introduction

1.1. The k-Young lattice

Recall that λ is a successor of a partition μ in the Young lattice when λ is obtained by adding an addable corner to μ where partitions are identified by their Ferrers diagrams, with rows weakly decreasing from bottom-to-top. This relation, which we denote "$\mu \to \lambda$", occurs naturally in the classical Pieri rule

$$h_1[X]\, s_\mu[X] = \sum_{\lambda:\, \mu\to\lambda} s_\lambda[X], \tag{1.1}$$

and the partial order of the Young lattice may be defined as the transitive closure of $\mu \to \lambda$. It was experimentally observed that the k-Schur functions [9,11] satisfy the rule

$$h_1[X]\, s_\mu^{(k)}[X] = \sum_{\lambda:\, \mu\to_k\lambda} s_\lambda^{(k)}[X]\,, \tag{1.2}$$

where "$\mu \to_k \lambda$" is a certain subrelation of "$\mu \to \lambda$". This given, the partial order of the k-Young lattice Y^k is defined as the transitive closure of $\mu \to_k \lambda$.