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Shifted products that are coprime pure powers

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Abstract

A set A of positive integers is called a coprime Diophantine powerset if the shifted product $ab + 1$ of two different elements a and b of A is always a pure power, and the occurring pure powers are all coprime. We prove that each coprime Diophantine powerset $A \subset \{1, \ldots, N\}$ has $|A| \leqslant 8000 \log N/\log\log N$ for sufficiently large N. The proof combines results from extremal graph theory with number theory. Assuming the famous *abc*-conjecture, we are able to both drop the coprimality condition and reduce the upper bound to $c \log\log N$ for a fixed constant c.

MSC: primary 11B75; 11D99; secondary 05D10; 05C38

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1. Introduction

A finite set A of integers is called a Diophantine n-tuple if $|A| = n$ and $ab + 1$ is a perfect square for all elements a and b of A with $a \neq b$. Diophantus of Alexandria studied such sets and found the following examples of rational numbers: $A = \{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$. Fermat found the following set of integers: $\{1, 3, 8, 120\}$. Euler found a parametric solution $\{a, b, a+b+2r, 4r(r+a)(r+b)\}$, where $ab+1 = r^2$. Baker and Davenport [1] proved that 120 is the only positive integer that extends the triple $\{1, 3, 8\}$ to a Diophantine quadruple. This implies that Fermat's example cannot be extended to a Diophantine quintuple. A well-known folklore conjecture asserts that there are no Diophantine 5-tuples. In this direction, Dujella proved that there are no Diophantine 6-tuples and that there are at most finitely many Diophantine quintuples [8,9]. Dujella maintains an interesting web page (see [10]) on this and related problems, giving many further references.

Recently, Bugeaud and Dujella [6] obtained a uniform upper bound of 7 for the cardinality of the set A when the set of squares is replaced by the set of kth powers of integers. A further generalization arises when, in addition, the exponent k is also allowed to vary. This leads to the following definition: we call a set A of positive integers a *Diophantine powerset* if $ab + 1$ is always a pure power for different elements a and b of A. In view of the aforementioned results, it is reasonable to conjecture that all Diophantine powersets are finite, their cardinality being bounded by an absolute constant. However, at present only the following weaker results are known. Gyarmati et al. ([18], see also [17], Theorem 6.4) showed that for sufficiently large N any Diophantine powerset $A \subset \{1, \ldots, N\}$ has cardinality

$$|A| < 340\frac{(\log N)^2}{\log\log N}, \tag{1}$$

so Diophantine powersets are very thin. More recently, Bugeaud and Gyarmati [7] obtained a slight improvement of this result, namely they proved

$$|A| \leqslant 177\,000(\log N/\log\log N)^2.$$

In their proof Gyarmati, Sárközy and Stewart defined for each k a graph where the vertices are the elements of A, and an edge connects the vertices a_i and a_j if and only if $a_i a_j + 1$ is a perfect kth power. Using that these graphs do not contain a cycle of length 4 they obtained (1). One may wonder whether a stronger bound can be proved by imposing a further, not too restrictive, condition on the set A. The purpose of this paper is to show that this is indeed the case.

We call a set A of positive integers a *coprime Diophantine powerset* if $ab + 1$ is always a pure power for different elements a and b of A, where in addition all occurring powers are coprime in pairs. This condition is not too restrictive since it includes the following important case. If the elements are multiples of $P = \prod_{p<y} p$, where the product is taken over primes less than y, then the numbers $a_i a_j + 1$ do not have any small prime $p \leqslant y$ as a common factor and therefore many of these might be coprime. But it is known that, for example, the most difficult case for giving an upper bound for the number of squares in arithmetic progressions is when the common difference is the product of many small