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Graphical condensation for enumerating perfect matchings☆

Weigen Yan[a,b], Fuji Zhang[c]

[a] School of Sciences, Jimei University, Xiamen 361021, P.R.China
[b] Institute of Mathematics, Academia Sinica, Nankang, Taipei, Taiwan 11529, R.O.China
[c] Department of Mathematics, Xiamen University, Xiamen 361005, P.R.China

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Abstract

The method of graphical condensation for enumerating perfect matchings was found by Propp (Theoret. Comput. Sci. 303 (2003) 267), and was generalized by Kuo (Theoret. Comput. Sci. 319 (2004) 29). In this paper, we obtain some more general results on graphical condensation than Kuo's. Our method is also different from Kuo's. As applications of our results, we obtain a new proof of Stanley's multivariate version of the Aztec diamond theorem and we enumerate perfect matchings of a type of molecular graph. Finally, a combinatorial identity on the number of plane partitions is also given.

Keywords: Graphical condensation; Perfect matching; Symmetric graph; Aztec diamond; Plane partition

1. Introduction

Throughout this paper, we suppose that $G = (V(G), E(G))$ is a simple graph with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and the edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$, if not specified. A perfect matching of G is a set of independent edges of G covering all vertices of G. We denote the set of perfect matchings of G by $\mathcal{M}(G)$ and the number of perfect matchings of G by $M(G)$. If G is a weighted graph, the weight of a perfect matching P of

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E-mail address: weigenyan@263.net

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G is defined to be the product of weights of edges in P. We also denote the sum of weights of perfect matchings of G by $M(G)$. Let A be a subset of the vertex set $V(G)$. By $G - A$ we denote the induced subgraph of G by deleting all vertices in A and the incident edges from G.

Zeilberger [15] gave a bijective proof of the Dodgson's determinant-evaluation rule:

$$\begin{aligned}&\det\left[(a_{ij})_{1\leqslant i\leqslant n}^{1\leqslant j\leqslant n}\right]\det\left[(a_{ij})_{2\leqslant i\leqslant n-1}^{2\leqslant j\leqslant n-1}\right]\\ &\quad=\det\left[(a_{ij})_{1\leqslant i\leqslant n-1}^{1\leqslant j\leqslant n-1}\right]\det\left[(a_{ij})_{2\leqslant i\leqslant n}^{2\leqslant j\leqslant n}\right]\\ &\qquad-\det\left[(a_{ij})_{1\leqslant i\leqslant n-1}^{2\leqslant j\leqslant n}\right]\det\left[(a_{ij})_{2\leqslant i\leqslant n}^{1\leqslant j\leqslant n-1}\right].\end{aligned}$$

Zeilberger's proof involves superimposing a perfect matching of one bipartite graph onto a perfect matching of another, and then partitioning that union into perfect matchings of two other bipartite graphs. Kuo [8] called this procedure, for enumerating perfect matchings, "graphical condensation". In terms of this idea, Propp [11] proved the following result.

Proposition 1.1 (*Propp [11]*). *Let $G = (U, V)$ be a plane bipartite graph in which $|U| = |V|$. Let vertices $a, b, c,$ and d form a 4-cycle face in G, $a, c \in U$, and $b, d \in V$. Then*

$$\begin{aligned}M(G)M(G-\{a,b,c,d\}) = {} & M(G-\{a,b\})M(G-\{c,d\})\\ & +M(G-\{a,d\})M(G-\{b,c\}).\end{aligned}$$

Kuo [8] generalized Propp's above result as follows.

Proposition 1.2 (*Kuo [8]*). *Let $G = (U, V)$ be a plane bipartite graph in which $|U| = |V|$. Let vertices $a, b, c,$ and d appear in a cyclic order on a face of G.*

(1) *If $a, c \in U$ and $b, d \in V$, then*

$$\begin{aligned}M(G)M(G-\{a,b,c,d\}) = {} & M(G-\{a,b\})M(G-\{c,d\})\\ & +M(G-\{a,d\})M(G-\{b,c\}).\end{aligned}$$

(2) *If $a, b \in U$ and $c, d \in V$, then*

$$\begin{aligned}M(G-\{a,d\})M(G-\{b,c\}) = {} & M(G)M(G-\{a,b,c,d\})\\ & +M(G-\{a,c\})M(G-\{b,d\}).\end{aligned}$$

In this paper, we use Ciucu's matching factorization theorem in [1] to prove some more general results on graphical condensation for enumerating perfect matchings than Kuo's. As applications of our results, we obtain a new proof of Stanley's multivariate version of the Aztec diamond theorem and we enumerate perfect matchings of a type of molecular graph. Finally, a combinatorial identity on the number of plane partitions is also given.