Available online at www.sciencedirect.com

SCIENCE DIRECT®

ELSEVIER

Journal of Combinatorial Theory, Series A 109 (2005) 1–23

Journal of Combinatorial Theory Series A

www.elsevier.com/locate/jcta

The embedding in AG$(3, q)$ of $(0, 2)$-geometries with no planar nets

Nikias De Feyter

Department of Pure Mathematics and Computer Algebra, Ghent University, Krijgslaan 281 - S22, B-9000 Gent, Belgium

Received 8 May 2003
Available online 12 November 2004

Abstract

This paper is part of the classification of $(0, \alpha)$-geometries $(\alpha > 1)$ embedded in AG(n, q). We study $(0, 2)$-geometries of order $(2^h - 1, t)$ embedded in AG$(3, 2^h)$ such that there are no planar nets. In the case $t \neq 2^h$, we prove some severe combinatorial restrictions. In the case $t = 2^h$ we provide a classification.

PACS: 05B25; 51E20

Keywords: $(0, \alpha)$-Geometry; Affine embedding; Full embedding; Planar oval set

1. Introduction

A $(0, \alpha)$-*geometry* $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a connected partial linear space of order (s, t) with the property that for every anti-flag (p, L) the number $\alpha(p, L)$ of lines of $\mathcal{S}$ through p intersecting L equals 0 or a constant α. If $\alpha(p, L) = \alpha$ for every anti-flag (p, L) then $\mathcal{S}$ is called a *partial geometry* pg(s, t, α) [1]. In this case the point graph of $\mathcal{S}$ is a strongly regular graph. A partial geometry pg$(s, t, 1)$ is called a *generalized quadrangle* [14] and a pg(s, t, t) is called a (*Bruck*) *net* of order $s + 1$ and degree $t + 1$. If a $(0, \alpha)$-geometry $\mathcal{S}$ has a strongly regular point graph then we call it a *semipartial geometry* spg(s, t, α, μ) [11]. Here μ is the number of vertices adjacent to two nonadjacent vertices in the point graph of $\mathcal{S}$.

E-mail address: ndfeyter@cage.ugent.ac.be

0097-3165/$ - see front matter © 2004 Published by Elsevier Inc.
doi:10.1016/j.jcta.2004.06.012

A $(0, \alpha)$-geometry $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ is said to be *fully embedded* (or, shortly, *embedded*) in an affine space $\mathrm{AG}(n, q)$ if the lines of $\mathcal{S}$ are lines of $\mathrm{AG}(n, q)$, if $\mathcal{P}$ is the set of all affine points on the lines of $\mathcal{S}$ and if I is as in $\mathrm{AG}(n, q)$. We also require that $\mathcal{P}$ spans $\mathrm{AG}(n, q)$. We say that $\mathcal{S}$ has a *planar net* if there is an affine plane such that the points and lines of $\mathcal{S}$ in it form a net. We will often identify a subspace of a projective or affine space with its point set. The space at infinity of $\mathrm{AG}(n, q)$ will be denoted by π_∞, while $\mathrm{PG}(n, q)$ will denote the projective completion of $\mathrm{AG}(n, q)$.

An important question in finite geometry is which geometries can be embedded in finite projective or affine spaces. This problem has been solved for various types of geometries. For example the embedding in $\mathrm{PG}(n, q)$ is solved for generalized quadrangles [2], for partial geometries [4] and for semipartial geometries, dual semipartial geometries and $(0, \alpha)$-geometries [5,16]. The embedding in $\mathrm{AG}(n, q)$ of generalized quadrangles and partial geometries is also solved [15]. About the embedding in $\mathrm{AG}(n, q)$ of semipartial geometries and $(0, \alpha)$-geometries only partial results are known. For example, the embedding in $\mathrm{AG}(2, q)$ and $\mathrm{AG}(3, q)$ of semipartial geometries is solved [10]. The embedding in $\mathrm{AG}(n, q)$, $n \geqslant 4$, of semipartial geometries is still an open problem. This paper is part of a series of papers in which we solve this problem. In fact, we give the answer to a more general question: which are the embeddings in $\mathrm{AG}(n, q)$ of $(0, \alpha)$-geometries?

Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be an incidence structure embedded in $\mathrm{AG}(n, q), n > 2$, and let U be a proper subspace of $\mathrm{AG}(n, q)$ of dimension at least 2. Then we define an incidence structure $\mathcal{S}_U = (\mathcal{P}_U, \mathcal{B}_U, \mathrm{I}_U)$, where $\mathcal{P}_U = \mathcal{P} \cap U$, $\mathcal{B}_U = \{L \in \mathcal{B} \mid L \subset U\}$ and I_U is the incidence I restricted to $\mathcal{P}_U$ and $\mathcal{B}_U$.

Lemma 1.1 (*De Clerck and Delanote [3]*). *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a $(0, \alpha)$-geometry, $\alpha > 1$, embedded in $\mathrm{AG}(n, q)$, $n \geqslant 3$, and let U be a proper subspace of $\mathrm{AG}(n, q)$ of dimension at least 2. Then every connected component of $\mathcal{S}_U$ which contains two intersecting lines is a $(0, \alpha)$-geometry.*

Lemma 1.1 is the main tool to study $(0, \alpha)$-geometries, $\alpha > 1$, embedded in $\mathrm{AG}(n, q)$. The method we use is as follows. First, the embeddings in lower-dimensional affine spaces are studied (typically for this kind of problems $n = 3$ is the hardest case). Then applying Lemma 1.1 the information obtained for lower dimensions is used to study embeddings in higher-dimensional affine spaces. Here, an induction argument on the dimension n is applied to obtain results for general n. Note that this method cannot be used to study affine embeddings of semipartial geometries because Lemma 1.1 does not hold for semipartial geometries. In this case connected components of $\mathcal{S}_U$ need not have strongly regular point graphs.

If $\mathcal{S}$ is a $(0, \alpha)$-geometry, $\alpha > 1$, embedded in $\mathrm{AG}(2, q)$ then it has been proved in [3] that either $\mathcal{S}$ is a Bruck net or a $\mathrm{pg}(q - 1, 1, 2)$. These are, respectively, the types III and IV planes in the following lemma.

Lemma 1.2 (*De Clerck and Delanote [3]*). *Let $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a $(0, \alpha)$-geometry, $\alpha > 1$, embedded in $\mathrm{AG}(n, q)$, $n > 2$, and let π be a plane of $\mathrm{AG}(n, q)$. Then π is of one of the following four types.*