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Contractible bonds in graphs

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Abstract

This paper addresses a problem posed by Oxley (Matroid Theory, Cambridge University Press, Cambridge, 1992) for matroids. We shall show that if G is a 2-connected graph which is not a multiple edge, and which has no K_5-minor, then G has two edge-disjoint non-trivial bonds B for which G/B is 2-connected.

MSC: 05C38; 05C40; 05C70

Keywords: Bond; Minor; Contractible

1. Introduction

For a graph G we shall let $\varepsilon(G)$ and $v(G)$ denote the number of edges and vertices in G, respectively. For a set of edges or vertices A of $V(G)$, we let $\mathbf{G}(\mathbf{A})$ denote the subgraph induced by A. For sets of vertices $X \subseteq V(G)$ and $Y \subseteq V(G)$ we denote the set of edges having one endpoint in X and the other in Y by $[\mathbf{X}, \mathbf{Y}]$. A *cutset* is a set of edges $[X, \overline{X}]$ for some X. A cutset which is minimal is called a *bond* or *cocycle*; that is, $B = [X, \overline{X}]$ is a bond if and only if both $G(X)$ and $G(\overline{X})$ are connected subgraphs. A bond B is said to be *trivial* if $B = [\{v\}, V(G)\backslash\{v\}]$ for some vertex v. A collection of edge-disjoint bonds of a graph which partitions its edges is called a *bond decomposition*. If in addition all its bonds are non-trivial, then the decomposition is said to be *non-trivial*.

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For $A \subset E(G)$ we let $\mathbf{G/A}$ denote the graph obtained by contracting the edges of A. For $v \in V(G/A)$ we denote by $> \mathbf{v} <_{\mathbf{A}}$ the vertices in the component of $G' = G(A) \cup V(G)$ corresponding to v. For an edge $e \in E(G/A)$ we let $> \mathbf{e} <_{\mathbf{A}}$ denote the corresponding edge in G. Similarly, for a subset of vertices (resp. edges) X of G/A we let $> \mathbf{X} <_{\mathbf{A}}$ denote the subset of vertices (resp. edges) $\bigcup_{x \in X} > x <_A$. For a subgraph H of G/H induced by $V(H)$ we let $> \mathbf{H} <_{\mathbf{A}}$ denote the subgraph of G induced by $> V(H) <_A$. For each vertex $v \in V(G)$ we associate the vertex $u \in V(G/A)$ where $v \in > u <_A$. We denote u by $\langle \mathbf{v} \rangle_{\mathbf{A}}$. Similarly, for an edge $e \in E(G) \backslash A$ we associate the edge $e' \in E(G/A)$ where $e = > e' <_A$. We denote e' by $\langle \mathbf{e} \rangle_{\mathbf{A}}$. For a subset of vertices $X \subseteq V(G)$ we let $\langle \mathbf{X} \rangle_{\mathbf{A}} = \{\langle v \rangle_A : v \in X\}$ and for a subset of edges $Y \subset E(G)$ we let $\langle \mathbf{Y} \rangle_{\mathbf{A}} = \{\langle e \rangle_A : e \in Y \backslash A\}$.

J. Oxley proposed the following problem in [7]:

1.1 Problem. *Let M be a simple connected binary matroid having cogirth at least* 4. *Does M have a circuit C such that* $M \backslash C$ *is connected*?

Here, by *cogirth* of a matroid M we mean the minimum cardinality of a cocircuit in M. For graphic matroids, this problem has been answered in the affirmative by a number of authors including Jackson [3], Mader [5], and Thomassen and Toft [8]. Recently, Goddyn and Jackson [1] proved that for any connected, binary matroid M having cogirth at least 5 which does not have either a F_7-minor or a F_7^*-minor, there is a cycle C for which $M \backslash C$ is connected. For cographic matroids, the above problem translates as follows. A circuit T in $M^*(G)$ corresponds to a bond in G. The matroid $M^*(G) \backslash T$ is connected if and only if either $|E(G/T)| = 1$ or G/T is loopless and 2-connected. Oxley's problem for cographic matroids can be restated as:

1.2 Problem. *Given G is a* 2-*connected*, 3-*edge connected graph with girth at least* 4, *does G contain a bond B such that* G/B *is* 2-*connected*?

We say that a collection of edges A in a 2-connected graph G is *contractible* if G/A is 2-connected. We say that a bond is *good* if it is both non-trivial and contractible. We call two edge-disjoint good bonds a *good pair* of bonds.

In [4], an example is given which shows that the answer to this problem is in general negative. The main result of this paper addresses Oxley's problem in the case of non-simple cographic matroids. Here there is a small example of a graph based on K_5 which has no contractible bonds: let B be a bond of cardinality 6 in K_5, and let G be the graph obtained from K_5 by duplicating each edge in $E(K_5) \backslash B$ and then subdividing both edges of each resulting digon exactly once (see Fig. 1). Then G is 2-connected with girth at least 4, but contracting any bond of G leaves a graph which is not 2-connected. We say that a digon is *isolated* if it is a multiple 2-edge consisting of two non-loop edges $\{e, f\}$ where no other edge has the same end vertices as e and f. In [2], the following theorem was proved which confirmed a conjecture of Jackson [3]:

1.3 Theorem. *Let G be a* 2-*connected graph having* $k \in \{0, 1\}$ *vertices of degree* 3 *and which has no Petersen graph minor and which is not a cycle. Then G has* $2 - k$ *edge-disjoint*