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Covering arrays on graphs

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Abstract

Two vectors v, w in $\mathbb{Z}_g^n$ are *qualitatively independent* if for all pairs $(a, b) \in \mathbb{Z}_g \times \mathbb{Z}_g$ there is a position i in the vectors where $(a, b)=(v_i, w_i)$. A covering array on a graph G, $CA(n, G, g)$, is a $|V(G)|\times n$ array on $\mathbb{Z}_g$ with the property that any two rows which correspond to adjacent vertices in G are qualitatively independent. The smallest possible n is denoted by $CAN(G, g)$. These are an extension of covering arrays. It is known that $CAN(K_{\omega(G)}, g) \leqslant CAN(G, g) \leqslant CAN(K_{\chi(G)}, g)$. The question we ask is, are there graphs with $CAN(G, g) < CAN(K_{\chi(G)}, g)$? We find an infinite family of graphs that satisfy this inequality. Further we define a family of graphs $QI(n, g)$ that have the property that there exists a $CAN(n, G, g)$ if and only if there is a homomorphism to $QI(n, g)$. Hence, the family of graphs $QI(n, g)$ defines a generalized colouring. For $QI(n, 2)$, we find a formula for both the chromatic and clique number and determine two necessary conditions for $CAN(G, 2) < CAN(K_{\chi(G)}, 2)$. We also find the cores of all the $QI(n, 2)$ and use this to prove that the rows of any covering array with $g = 2$ can be assumed to have the same number of 1's.

Keywords: Orthogonal array; Covering array; Software and network testing; Graph homomorphism; Core; Generalized colourings

1. Introduction

Covering arrays, also known as *qualitatively t-independent families* of vectors or as *t-surjective arrays* have been widely studied. They are generalizations of both orthogonal arrays and Sperner systems. Bounds and constructions of covering arrays have been derived from algebra, set systems, intersecting codes, design theory and Sperner systems

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$$\begin{array}{ccccc} 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{array}$$

Fig. 1. The optimal covering array $CA(5, 4, 2)$.

[1,11,16,19,21,22]. Covering arrays have industrial applications to software and circuit testing, switching networks, drug screening and data compression; they also have mathematical applications to the construction of difference matrices, search theory and truth functions. [2–4,10,12,14,15,20,23].

In this paper, we extend the definition of a covering array to include a graph structure. This is an open problem in the conclusion of the second author's thesis [18]. The binary case of this problem has been studied by Seroussi and Bshouty who proved that determining the existence of an optimal binary covering array on a graph is an NP-complete problem [15].

To start we define t-qualitatively independence and strength-t covering arrays.

Definition 1 (*t-Qualitative independence*). A set of vectors with entries from $\mathbb{Z}_g$ are *t-qualitatively independent* if for any t-subset, $\{v_i\}$, of vectors and any ordered t-tuple of elements $(g_1, g_2, \ldots, g_t) \in \mathbb{Z}_g^t$ there exists a j such that for each vector v_i the jth coordinate $v_{ij} = g_i$.

Definition 2 (*Covering array*). A *t-covering array* with alphabet size g, k rows and size n is a $k \times n$ array on $\mathbb{Z}_g$ with the property that any set of t rows is t-qualitatively independent. This is denoted by $t - CA(n, k, g)$.

This is the standard covering array that is considered in most literature on the subject. In this paper, only 2-covering arrays are considered, they will simply be called covering arrays and denoted $CA(n, k, g)$. Similarly, any pair of 2-qualitatively independent vectors will simply be called qualitatively independent.

The smallest possible size of a covering array is denoted

$$CAN(k, g) = \min_{l \in \mathbb{N}} \{l : \exists\, CA(l, k, g)\}.$$

Example 1. An example of a covering array $CA(5, 4, 2)$ is shown in Fig. 1.

In testing applications, each row in the array represents a particular component of the system being tested. It may be an input variable, a network node, a subroutine or a hardware component. Each column in the array corresponds to a test on the system. The goal is to produce an array with the fewest number of columns, hence tests. Strength two covering arrays test all pairwise interactions. This requires far fewer tests than complete testing but in practice provides good test coverage [3,6].