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Quantum cohomology via D-modules

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Dedicated to Graeme Segal on the occasion of his 60th birthday

Abstract

We propose a new point of view on quantum cohomology, motivated by the work of Givental and Dubrovin, but closer to differential geometry than the existing approaches. The central object is a D-module which "quantizes" a commutative algebra associated to the (uncompactified) space of rational curves. Under appropriate conditions, we show that the associated flat connection may be gauged to the flat connection underlying quantum cohomology. This method clarifies the role of the Birkhoff factorization in the "mirror transformation", and it gives a new algorithm (requiring construction of a Groebner basis and solution of a system of o.d.e.) for computation of the quantum product.

Keywords: Quantum cohomology; D-module; Birkhoff factorization

0. Introduction

Quantum cohomology first arose in physics, and its (mathematically conjectural) properties were supported by physical intuition. A rigorous mathematical definition came later, based on deep properties of certain moduli spaces. We shall propose another point of view on quantum cohomology, closer in spirit to differential geometry.

The main ingredient in our approach is a flat connection, considered as a holonomic D-module (or maximally overdetermined system of p.d.e.). This object itself is not new: Givental's "quantum cohomology D-module" is already well known [10], and the associated flat connection appears in Dubrovin's

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theory of Frobenius manifolds [7]. But, in the existing literature, the D-module plays a subservient role, being a consequence of the construction of the Gromov–Witten invariants and the quantum cohomology algebra. For us, the D-module will be the main object of interest.

We define a quantization of a (commutative) algebra $\mathscr{A}$ to be a (non-commutative) D-module M^h which satisfies certain properties. The quantum cohomology D-module is a particular kind of quantization, which arises in the following way. For a Kähler manifold M, we start with an algebra $\mathscr{A}$ which is associated to the "raw data" consisting of the set of all rational curves in M. Then we construct (or assume the existence of) a quantization M^h. Next we transform M^h into a new D-module $\hat{M}^h$ with certain properties. Finally, de-quantization ("semi-classical limit") produces a commutative algebra $\hat{\mathscr{A}}$, which (under appropriate conditions) turns out to be the quantum cohomology QH^*M.

Our scope will be very modest in this article: we consider only the "small" quantum cohomology algebra QH^*M of a manifold M whose ordinary cohomology algebra H^*M is generated by two-dimensional classes. But this case is sufficiently non-trivial to demonstrate that our method has something to offer, both conceptually and computationally. The most obvious conceptual benefit is that the usual moduli space $\mathscr{M}$ has been replaced by the D-module M^h. As a first application we give an algorithm for computing the structure constants of the quantum cohomology algebra (3-point genus zero Gromov–Witten invariants), in the case of a Fano manifold. This involves a Gröbner basis calculation and a finite number of "quadratures"; it is quite different from previously known methods. A second application is a new interpretation of the "mirror coordinate transformation". Impressively mysterious in its original context [12,13,21–23], it arises here in a straightforward differential geometric fashion, reminiscent of the well-known transformation to local Euclidean coordinates for a flat Riemannian manifold.

Here is a more detailed description of the organization of this paper. In Section 0 we review some facts concerning D-modules, mainly to establish the notation. In Section 2 we recall the quantum cohomology algebra and the quantum product, again to set up the notation. "Quantum cohomology algebra" refers to the isomorphism type of the algebra, while "quantum product" means the product operation on the vector space H^*M, i.e. a way of multiplying ordinary cohomology classes.

Our point of view is introduced in Section 3: we start with an algebra $\mathscr{A}$ and construct from it both a "quantum cohomology algebra" and a "quantum product". The method is conceptually straightforward. To a quantization M^h of $\mathscr{A}$ there corresponds a flat connection $\nabla = d + \Omega^h$, where Ω^h has a simple pole at $h = 0$. We may write $\Omega^h = L^{-1}\,\mathrm{d}L$ for some loop group-valued map L. Replacing L by L_-, where $L = L_- L_+$ is the Birkhoff factorization, we obtain $\hat{\Omega}^h = L_-^{-1}\,\mathrm{d}L_-$, and the connection $d + \hat{\Omega}^h$ is the required connection. The map L is a generating function for certain Gromov–Witten invariants but we shall not need it. Our main interest is the gauge transformation $L_+ = Q_0 + O(h)$ which converts Ω^h to $\hat{\Omega}^h$. For the manifolds discussed here, $\mathscr{A}$ and M^h are known, and Ω^h can be computed. If L_+ can be computed, then $\hat{\Omega}^h$ (and the quantum cohomology algebra, together with its structure constants) can be computed too.

In Section 4 we discuss the case of Fano manifolds. Here it turns out that $\mathscr{A} = \hat{\mathscr{A}}$, i.e. the "provisional" algebra is actually the "correct answer". The gauge transformation L_+ has a special form but it is not trivial; indeed, its first term Q_0 tells us how to produce the quantum product. Thus all quantum products can be *determined explicitly* by our method from the relations of the quantum cohomology algebra (more precisely, from their quantizations). The following two families of manifolds are of special interest:

(1) Let $M = G/B$, the full flag manifold of a complex semi-simple Lie group G. The quantum cohomology algebra was found originally by Givental and Kim [14,19] and justified via the conventional