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Spaces embeddable as regular closed subsets into *acc* spaces and (a)-spaces

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Abstract

It is well known that some classes of spaces such as absolutely countably compact (abbreviated *acc*) spaces and (a)-spaces are not hereditary with respect to closed, and even regular closed subspaces. In this paper, we investigate conditions for spaces being regular closed embeddable into spaces with certain covering properties, in particular we characterize spaces which can be embedded as regular closed subsets into *acc* spaces and (a)-spaces. Some examples are presented as applications of the criteria. Two problems raised by Matveev [Topology Appl. 80 (1997) 169–175] are answered.

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1. Introduction

Matveev [4,5] introduced *acc* spaces ((a)-spaces) as those spaces X in which for every open cover $\mathscr{U}$ of X and every dense subset $D \subset X$ there exists a finite (discrete and closed in X) subset $F \subset D$ such that

$$\mathrm{St}(F, \mathscr{U}) = \bigcup\{U \in \mathscr{U}\colon\ U \cap F \neq \emptyset\} = X.$$

It is known that in many ways (a)-spaces behave like normal spaces [5]. However, unlike normal spaces, (a)-spaces are not a hereditary class with respect to closed subspaces. So it is natural to ask what kind of spaces may be represented as certain type of closed subspaces of (a)-spaces. In [6], Matveev proved that every Tychonoff space can be embedded as a nowhere dense zero-set into a Tychonoff (a)-space, and every Tychonoff countably compact space can be embedded as a nowhere dense zero-set into an *acc* space; also he noticed that a regular closed G_δ set in an *acc* space need not to be *acc*. So Matveev raised the following questions:

Question 1 [6,7]. Characterize those Tychonoff countably compact spaces which can be embedded as regular closed subsets into Tychonoff *acc* spaces.

Question 2 [6]. Characterize those Tychonoff spaces which can be embedded as regular closed subsets into Tychonoff (a)-spaces.

In this paper, we give the solutions to the two questions. In Section 2, we define spaces with certain covering properties and show necessary conditions and sufficient conditions required for a space to be embedded as a regular closed subset into these spaces. Then in Section 3, using these conditions, we get Theorem 4 which characterize the spaces which can be embedded as regular closed subsets into *acc* spaces and (a)-spaces. Corollaries 5 and 6 answer Questions 1 and 2 respectively. Some "positive" applications of our criteria are presented (see Corollary 8 and Example 2). In Section 4, we present some constructions which can be useful in producing (a)- and *acc*-extensions with certain properties. In the last section, we give some examples of spaces which cannot be embedded into an (a)-space or an *acc* space by means of our criteria.

Recall that a space X is *countably compact* if every countable open cover of X has a finite subcover. A characterization of countable compactness states that a Hausdorff space X is countably compact if and only if for every open cover $\mathscr{U}$ of X there exists a finite subset F of X such that $\mathrm{St}(F, \mathscr{U}) = X$ ([1], [2, Problems 3.12.23(d)]). A space X is *star-Lindelöf* (*absolutely star-Lindelöf*) if for every open cover $\mathscr{U}$ of X (and every dense subset D of X) there exists a countable subset F of X (of D) such that $\mathrm{St}(F, \mathscr{U}) = X$ (see [7]). It is easy to see that in countably compact spaces, (a)-property, *acc*-property and absolute star-lindelöfness coincide.

Throughout the paper, by a space we mean a topological space. A subset E of a space X is said to be regular closed if $E = \overline{\mathrm{int}\, E}$. $\mathbb{R}$ is the set of all real numbers. The cardinality of a set A is denoted by $|A|$. For a cardinal κ, κ^+ denotes the smallest cardinal greater than κ. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals.