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Remainders in compactifications and generalized metrizability properties

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Abstract

When does a Tychonoff space X have a Hausdorff compactification with the remainder belonging to a given class of spaces? A classical theorem of Henriksen and Isbell and certain theorems, involving a new completeness type property introduced below, are applied to obtain new results on remainders of topological spaces and groups. In particular, some strong necessary conditions for a topological group to have a metrizable remainder, or a paracompact p-remainder, are established (the group itself turns out to be a paracompact p-space (Theorem 4.8)). It follows that if a non-locally compact topological group G is metrizable at infinity, then G is a Lindelöf p-space, and the Souslin number of G is countable (Corollary 4.10). This solves Problem 10.28 from [M. Hušek, J. van Mill (Eds.), Recent Progress in General Topology, vol. 2, North-Holland, 2002, pp. 1–57].

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1. Introduction

When does a Tychonoff space X have a Hausdorff compactification with the remainder belonging to a given class of spaces? A classical non-trivial result in this direction is the

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following theorem of M. Henriksen and J. Isbell: A Tychonoff space X is of countable type if and only if the remainder in any (or in some) Hausdorff compactification of X is Lindelöf [9]. Recall that a space X is of *countable type* if every compact subspace P of X is contained in a compact subspace $F \subset X$ that has a countable base of open neighbourhoods in X [1]. All metrizable spaces and all locally compact Hausdorff spaces are of countable type. Therefore, it follows from the theorem of Henriksen and Isbell that every remainder of a metrizable space is Lindelöf, and hence, is paracompact.

Notice, how two very different properties are brought together in Henriksen–Isbell's theorem, in a natural duality. Amazingly, we do not know much beyond this result. For example, we do not know when a Tychonoff space X has a Hausdorff compactification with a metrizable remainder, we do not know when there exists a compactification for X with a paracompact remainder or with a paracompact p-remainder, and so on. In fact, we do not know much about the relationship between properties of X and properties of remainders of X in Hausdorff compactifications, and whatever we know in this direction is very unsystematic.

Below we consider spaces whose remainders are close, in some sense, to being metrizable. In particular, we consider when a space has a remainder with a G_δ-diagonal. The strongest results are obtained for topological groups. Among the main results are Corollary 3.7, Theorems 3.3, 4.3 and 4.5, Corollary 4.11, Example 4.15, Theorems 4.6, 4.8, 4.14 and 4.19. Curiously, every remainder of a Lindelöf p-space is a Lindelöf p-space (Theorem 2.1). However, this statement does not generalize to paracompact p-spaces: the remainders of such spaces need not be paracompact p-spaces. However, we establish that if a topological group G has a remainder that is a paracompact p-space, then G itself is a paracompact p-space (Theorem 4.8). It follows from Theorems 4.6 and 2.1 that the remainder, in this case, is a Lindelöf p-space. Several new open problems are posed. One of them is to characterize non-locally compact topological groups that have a metrizable remainder.

"A space" in this article stands for a Tychonoff topological space. *A remainder* of a space X is a space $bX \setminus X$, where bX is a compactification of X. We say that a space X has a property $\mathcal{P}$ *at infinity* if some remainder of X has the property $\mathcal{P}$. Very often this implies that every remainder of X has the property $\mathcal{P}$. For example, this is the case if $\mathcal{P}$ is paracompactness. If γ is a family of subsets of a space X, and $x \in X$, then $St_\gamma(x) = \bigcup\{U \in \gamma\colon x \in U\}$. Recall that *paracompact p-spaces* are preimages of metrizable spaces under perfect mappings. A mapping is said to be *perfect* if it is continuous, closed, and all fibers are compact. *A Lindelöf p-space* is a preimage of a separable metrizable space under a perfect mapping. In general, we follow [7] in terminology and notation.

2. Remainders and Lindelöf p-spaces

Clearly, every separable metrizable space has a separable metrizable remainder. Here is a parallel result, curious and useful, though not difficult to prove.

Theorem 2.1. *If X is a Lindelöf p-space, then any remainder of X is a Lindelöf p-space.*