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Common stabilizations of Heegaard splittings of link exteriors ☆

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Abstract

We give a condition for a pair of unknotting tunnels of a non-trivial tunnel number one link to give a genus three Heegaard splitting of the link complement and show that every 2-bridge link has such a pair of unknotting tunnels.

MSC: 57M99

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1. Introduction

A *compression body* H is a 3-manifold obtained from a compact connected closed surface S by attaching 2-handles to $S \times I$ on $S \times \{1\}$ and capping off any resulting 2-sphere boundary components with 3-handles. $S \times \{0\}$ is denoted by $\partial_+ H$ and $\partial H - \partial_+ H$ is denoted by $\partial_- H$. A compression body H is called a *handlebody* if $\partial_- H = \emptyset$.

If a compact 3-manifold M is the union of two compression bodies H_1 and H_2 along their common "plus" boundary $S = \partial_+ H_1 = \partial_+ H_2$, we call the decomposition $M = H_1 \cup_S$

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H_2 a *Heegaard splitting* of M and S a *Heegaard surface* of M. The minimum number of the genus of S among all Heegaard splittings of M is called the *Heegaard genus* (or *genus*) of M.

Suppose $H_1 \cup_S H_2$ is a Heegaard splitting of a 3-manifold M and α is a properly embedded arc in H_2 parallel to an arc in S. That is, there is an embedded disk D in H_2 whose boundary is the union of α and an arc in $\partial_+ H_2$. Now add a neighborhood of α to H_1 and delete it from H_2. Once again the result is a Heegaard splitting $H_1' \cup_{S'} H_2'$, where the genus of each H_i' is one greater than H_i. This process is called a *stabilization* of S.

Every compact 3-manifold can be triangulated and any two triangulations of a 3-manifold are PL-equivalent [1,7]. It follows that every compact 3-manifold has a Heegaard splitting and any two Heegaard splittings of a 3-manifold have a common stabilization. In fact, there is no example of distinct Heegaard splittings of a same closed 3-manifold which cannot be made isotopic by a single stabilization of one of the splittings, and sufficient stabilizations of the other to ensure that the genus of the two surfaces is the same. This makes the following conjecture very optimistic.

Conjecture 1.1 [9]. *Suppose $H_1 \cup_S H_2$ and $H_1' \cup_{S'} H_2'$ are Heegaard splittings of the same 3-manifold of, genus $g \leqslant g'$ respectively. Then the splittings obtained by one stabilization of S' and $g' - g + 1$ stabilizations of S are isotopic.*

A *tunnel system* (or *tunnels*) of a knot or a link K is a collection of disjoint embedded arcs $t_1, t_2, \ldots, t_n$ in S^3 with $K \cap \bigcup_{i=1}^n t_i = \bigcup_{i=1}^n \partial t_i$ such that $H = \overline{S^3 - N(K \cup \bigcup_{i=1}^n t_i)}$ is a genus $n+1$ handlebody. (Here $N(X)$ denotes a regular neighborhood of X.) The tunnel system gives rise to a Heegaard splitting of the exterior of K

$$\overline{S^3 - N(K)} = H \cup_{\partial H} \overline{N\left(K \cup \bigcup_{i=1}^n t_i\right) - N(K)}$$

where $N(K)$ is contained in the interior of $N(K \cup \bigcup_{i=1}^n t_i)$. The minimum of such number n is called the *tunnel number* of K. If the tunnel number of K is 1, the tunnel is called an *unknotting tunnel* of K.

For a tunnel number one knot K, we consider two non-isotopic unknotting tunnels t_1, t_2 and corresponding Heegaard surfaces S_1, S_2 of the exterior of K. Now suppose $H = \overline{S^3 - N(K \cup t_1 \cup t_2)}$ is a genus three handlebody. This means that $S' = \partial H$ becomes a Heegaard surface for the genus two handlebodies $\overline{S^3 - N(K \cup t_1)}$ and $\overline{S^3 - N(K \cup t_2)}$. By [10], there is at most one Heegaard splitting of a handlebody of a given genus. This implies S' is a common stabilization of S_1 and S_2 and shows a validity of Conjecture 1.1. There are examples of knots having this property—torus knots and 2-bridge knots.

In this paper, we give a sufficient condition for tunnel number one links to have this property and show that 2-bridge links satisfy this condition.