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The representability number of a chain

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Abstract

For each pair of linear orderings (L, M), the representability number $\mathrm{repr}_M(L)$ of L in M is the least ordinal α such that L can be order-embedded into the lexicographic power $M^{\alpha}_{\mathrm{lex}}$. The case $M = \mathbb{R}$ is relevant to utility theory. The main results in this paper are as follows. (i) If κ is a regular cardinal that is not order-embeddable in M, then $\mathrm{repr}_M(\kappa) = \kappa$; as a consequence, $\mathrm{repr}_{\mathbb{R}}(\kappa) = \kappa$ for each $\kappa \geqslant \omega_1$. (ii) If M is an uncountable linear ordering with the property that $A \times_{\mathrm{lex}} 2$ is not order-embeddable in M for each uncountable $A \subseteq M$, then $\mathrm{repr}_M(M^{\alpha}_{\mathrm{lex}}) = \alpha$ for any ordinal α; in particular, $\mathrm{repr}_{\mathbb{R}}(\mathbb{R}^{\alpha}_{\mathrm{lex}}) = \alpha$. (iii) If L is either an Aronszajn line or a Souslin line, then $\mathrm{repr}_{\mathbb{R}}(L) = \omega_1$.

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1. Introduction

In this paper we deal with representations of linear orderings (also called chains) in ways that are useful in the field of mathematical economics called *utility theory* (see [6] for an overview of this topic). A key notion in utility theory is that of representability: a chain $(L, \prec)$ is *representable* (in $\mathbb{R}$) if there exists a map $u : L \to \mathbb{R}$, called a *utility function*,

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which is an order-embedding (i.e., $x \prec y$ if and only if $u(x) < u(y)$ for all $x, y \in L$). If we interpret $x \prec y$ as "y is preferred to x", then a utility function on L measures preferences quantitatively. In the traditional approach much attention has been given to characterizations of representable chains. A well-known result in this sense is the following (see, e.g., [2]). (Recall that a *jump* in a chain L is a pair $(a, b) \in L^2$ such that $a \prec b$ and the open interval (a, b) is empty.)

Theorem 1.1. *A chain is representable* (*in* $\mathbb{R}$) *if and only if it is separable in the order topology and has at most countably many jumps.*

A more recent approach to the problem of representability focuses on finding structural obstructions to the representability of a chain among its subchains (see [1,3]). Classical examples of chains for which representability fails are the real plane endowed with the lexicographic order $\mathbb{R}^2_{\text{lex}}$, the first uncountable ordinal ω_1 and its reverse ordering $\omega_1{}^*$. Recall that a chain L is *short* if neither ω_1 nor $\omega_1{}^*$ order-embed into L, and it is *long* otherwise; further, an *Aronszajn line* is an uncountable chain that is short and does not contain any uncountable representable subchain. The next result (from [1]) gives a subordering characterization of non-representable chains.

Theorem 1.2. *A chain L is non-representable* (*in* $\mathbb{R}$) *if and only if* (i) *it is long, or* (ii) *it order-embeds a non-representable subchain of the lexicographic plane, or* (iii) *it order-embeds an Aronszajn line.*

Our objective is to give a more descriptive classification of non-representable chains (and, more generally, of all chains). In this paper we begin to pursue this goal by classifying chains according to a measure of their "lexicographic complexity". To this aim we take the point of view that a chain which can be order-embedded in the lexicographically ordered real plane is representable, even if in a weaker sense. Such an ordering is realized in a way that is more complex than for suborderings of $\mathbb{R}$, but which still fits within the general utility concept. This is based on the observation that an order-embedding of $(L, \prec)$ into $\mathbb{R}^2_{\text{lex}}$ corresponds to two functions $u_1, u_2 : L \to \mathbb{R}$ with the property that for all $x, y \in L$, we have $x \prec y$ if and only if either $u_1(x) < u_1(y)$, or $u_1(x) = u_1(y)$ and $u_2(x) < u_2(y)$. In other words, preference in the sense of L corresponds to preference according to u_1 and u_2 together, but with u_1 being given higher priority.

More generally, we say that a chain $(L, \prec)$ is *α-representable* (in $\mathbb{R}$) if it can be order-embedded into the lexicographic power $\mathbb{R}^\alpha_{\text{lex}}$, where α is an ordinal number. This corresponds to having a representation of the preference ordering $\prec$ by a well-ordered family of utility functions $u_\xi : L \to \mathbb{R}$ indexed by the ordinals $\xi < \alpha$; for any $x, y \in L$ one has $x \prec y$ if and only if $u_\beta(x) < u_\beta(y)$ holds, where β is the least ordinal below α at which $u_\beta(x)$ and $u_\beta(y)$ differ. One can think of the ordinal indices as determining the relative importance of the utility functions u_ξ.

The least ordinal α for which a chain L is α-representable is called the *representability number of L* (in $\mathbb{R}$). More generally, for any pair of chains (L, M), we define the *representability number of L in M* as the least ordinal α such that L can be order-embedded into M^α_{lex}; this ordinal is denoted by $\text{repr}_M(L)$. In this paper we determine $\text{repr}_M(L)$ for