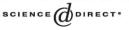


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The representability number of a chain

Alfio Giarlotta

Department of Mathematics, University of Illinois, Urbana, IL 61801, USA Department of Economics and Quantitative Methods, University of Catania, Catania 95129, Italy

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Abstract

For each pair of linear orderings (L, M), the representability number $\operatorname{repr}_M(L)$ of L in M is the least ordinal α such that L can be order-embedded into the lexicographic power M_{lex}^{α} . The case $M = \mathbb{R}$ is relevant to utility theory. The main results in this paper are as follows. (i) If κ is a regular cardinal that is not order-embeddable in M, then $\operatorname{repr}_M(\kappa) = \kappa$; as a consequence, $\operatorname{repr}_{\mathbb{R}}(\kappa) = \kappa$ for each $\kappa \ge \omega_1$. (ii) If M is an uncountable linear ordering with the property that $A \times_{\text{lex}} 2$ is not order-embeddable in M for each uncountable $A \subseteq M$, then $\operatorname{repr}_M(M_{\text{lex}}^{\alpha}) = \alpha$ for any ordinal α ; in particular, $\operatorname{repr}_{\mathbb{R}}(\mathbb{R}_{\text{lex}}^{\alpha}) = \alpha$. (iii) If L is either an Aronszajn line or a Souslin line, then $\operatorname{repr}_{\mathbb{R}}(L) = \omega_1$.

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1. Introduction

In this paper we deal with representations of linear orderings (also called chains) in ways that are useful in the field of mathematical economics called *utility theory* (see [6] for an overview of this topic). A key notion in utility theory is that of representability: a chain (L, \prec) is *representable* (in \mathbb{R}) if there exists a map $u: L \to \mathbb{R}$, called a *utility function*,

E-mail address: giarlott@math.uiuc.edu (A. Giarlotta).

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which is an order-embedding (i.e., $x \prec y$ if and only if u(x) < u(y) for all $x, y \in L$). If we interpret $x \prec y$ as "y is preferred to x", then a utility function on L measures preferences quantitatively. In the traditional approach much attention has been given to characterizations of representable chains. A well-known result in this sense is the following (see, e.g., [2]). (Recall that a *jump* in a chain L is a pair $(a, b) \in L^2$ such that $a \prec b$ and the open interval (a, b) is empty.)

Theorem 1.1. A chain is representable (in \mathbb{R}) if and only if it is separable in the order topology and has at most countably many jumps.

A more recent approach to the problem of representability focuses on finding structural obstructions to the representability of a chain among its subchains (see [1,3]). Classical examples of chains for which representability fails are the real plane endowed with the lexicographic order $\mathbb{R}^2_{\text{lex}}$, the first uncountable ordinal ω_1 and its reverse ordering ω_1^* . Recall that a chain *L* is *short* if neither ω_1 nor ω_1^* order-embed into *L*, and it is *long* otherwise; further, an *Aronszajn line* is an uncountable chain that is short and does not contain any uncountable representable subchain. The next result (from [1]) gives a subordering characterization of non-representable chains.

Theorem 1.2. A chain L is non-representable (in \mathbb{R}) if and only if (i) it is long, or (ii) it order-embeds a non-representable subchain of the lexicographic plane, or (iii) it order-embeds an Aronszajn line.

Our objective is to give a more descriptive classification of non-representable chains (and, more generally, of all chains). In this paper we begin to pursue this goal by classifying chains according to a measure of their "lexicographic complexity". To this aim we take the point of view that a chain which can be order-embedded in the lexicographically ordered real plane is representable, even if in a weaker sense. Such an ordering is realized in a way that is more complex than for suborderings of \mathbb{R} , but which still fits within the general utility concept. This is based on the observation that an order-embedding of (L, \prec) into $\mathbb{R}^2_{\text{lex}}$ corresponds to two functions $u_1, u_2: L \to \mathbb{R}$ with the property that for all $x, y \in L$, we have $x \prec y$ if and only if either $u_1(x) < u_1(y)$, or $u_1(x) = u_1(y)$ and $u_2(x) < u_2(y)$. In other words, preference in the sense of *L* corresponds to preference according to u_1 and u_2 together, but with u_1 being given higher priority.

More generally, we say that a chain (L, \prec) is α -representable (in \mathbb{R}) if it can be order-embedded into the lexicographic power $\mathbb{R}_{lex}^{\alpha}$, where α is an ordinal number. This corresponds to having a representation of the preference ordering \prec by a well-ordered family of utility functions $u_{\xi}: L \to \mathbb{R}$ indexed by the ordinals $\xi < \alpha$; for any $x, y \in L$ one has $x \prec y$ if and only if $u_{\beta}(x) < u_{\beta}(y)$ holds, where β is the least ordinal below α at which $u_{\beta}(x)$ and $u_{\beta}(y)$ differ. One can think of the ordinal indices as determining the relative importance of the utility functions u_{ξ} .

The least ordinal α for which a chain L is α -representable is called the *representability* number of L (in \mathbb{R}). More generally, for any pair of chains (L, M), we define the *representability number of* L in M as the least ordinal α such that L can be order-embedded into M_{lex}^{α} ; this ordinal is denoted by $\text{repr}_{M}(L)$. In this paper we determine $\text{repr}_{M}(L)$ for

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