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Span, chainability and the continua $\mathbb{H}^*$ and $\mathbb{I}_u$

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Dedicated to the memory of Zoli Balogh

Abstract

We show that the continua $\mathbb{I}_u$ and $\mathbb{H}^*$ are nonchainable and have span nonzero. Under CH this can be strengthened to surjective symmetric span nonzero.

We discuss the logical consequences of this.

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1. Introduction

Chainable (or arc-like) continua are ‘long and thin’; in an attempt to capture this idea in metric terms Lelek introduced, in [6], the notion of span. Chainable continua have span zero, which is useful in proving that certain continua are not chainable. The converse, a conjecture by Lelek in [7], is one of the main open problems in continuum theory today. While the particular value of the span of a continuum depends on the metric chosen, the distinction between span zero and span nonzero is a topological one. As chainability is a topological notion as well, Lelek’s theorem and conjecture are meaningful in the class of

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all Hausdorff continua. We investigate the chainability and span of several continua that are closely connected to the Čech–Stone compactification of the real line.

2. Preliminaries

2.1. Various kinds of span

The kinds of span that we consider in this paper are, in the metric case, defined as suprema of distances between the diagonal of the continuum and certain subcontinua of the square. The following families of subcontinua feature in these definitions:

$S(X)$: the symmetric subcontinua of X^2, i.e., those that satisfy $Z = Z^{-1}$;
$\Sigma(X)$: the subcontinua of X^2 that satisfy $\pi_1[Z] = \pi_2[Z]$; and
$\Sigma_0(X)$: the subcontinua of X^2 that satisfy $\pi_2[Z] \subseteq \pi_1[Z]$.

Here, π_1 and π_2 are the projections onto the first and second coordinates, respectively. It is clear that $S(X) \subseteq \Sigma(X) \subseteq \Sigma_0(X)$ and hence that $s(X) \leqslant \sigma(X) \leqslant \sigma_0(X)$, where

(1) $s(X) = \sup\{d(\Delta(X), Z)\colon\ Z \in S(X)\}$;
(2) $\sigma(X) = \sup\{d(\Delta(X), Z)\colon\ Z \in \Sigma(X)\}$; and
(3) $\sigma_0(X) = \sup\{d(\Delta(X), Z)\colon\ Z \in \Sigma_0(X)\}$.

These numbers are, respectively, the *symmetric span*, the *span* and the *semi-span* of X.

If one uses, in each definition, only the continua Z with $\pi_1[Z] = X$ then one gets the *surjective symmetric span*, $s^*(X)$, the *surjective span*, $\sigma^*(X)$, and the *surjective semi-span*, $\sigma_0^*(X)$, of X, respectively. The following diagram shows the obvious relationships between the six kinds of span.

$$\begin{array}{ccccc} s(X) & \longrightarrow & \sigma(X) & \longrightarrow & \sigma_0(X) \\ \uparrow & & \uparrow & & \uparrow \\ s^*(X) & \longrightarrow & \sigma^*(X) & \longrightarrow & \sigma_0^*(X) \end{array} \tag{1}$$

Topologically we can only distinguish between a span being zero or nonzero. A span is zero if and only if every continuum from its defining family intersects the diagonal. This defines span zero (or span nonzero) for the six possible types of span in general continua.

Below we will show that for the continua $\mathbb{H}^*$ and $\mathbb{I}_u$ all six kinds of span are nonzero. Diagram (1) shows that it will be most difficult to show that s^* is nonzero (or dually that it would be hardest to show that σ_0 is zero). Indeed, we will give successively more difficult proofs that the various spans are nonzero, where we traverse the diagram from top right to bottom left.

The need for these different proofs lies in their set-theoretic assumptions. We need nothing beyond ZFC to show that $\sigma^*(\mathbb{H}^*)$ and $\sigma(\mathbb{I}_u)$ are nonzero; to show that the other spans (in particular s^*) are nonzero we shall need the Continuum Hypothesis (CH).