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Lattices on Benson–Gordon type solvable Lie groups

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Abstract

Benson and Gordon constructed the unimodular solvable Lie group G^{BG}. In this paper, we prove that G^{BG} admits lattices. In addition, we construct compact symplectic solvmanifolds without the Hard Lefschetz property.

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Introduction

The purpose in this paper is to prove that the completely solvable Lie group constructed by Benson and Gordon admits a lattice, that is, a discrete co-compact subgroup. Benson and Gordon [2] constructed a completely Lie algebra $\mathfrak{g}^{\mathrm{BG}}$ given by

$$\mathfrak{g}^{\mathrm{BG}} = \mathrm{span}\{A, X_1, Y_1, Z_1, X_2, Y_2, Z_2\},$$
$$[X_1, Y_1] = Z_1, \qquad [X_2, Y_2] = Z_2,$$
$$[A, X_1] = X_1, \qquad [A, Y_1] = -2Y_1, \qquad [A, Z_1] = -Z_1,$$

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$$[A, X_2] = -X_2, \qquad [A, Y_2] = 2Y_2, \qquad [A, Z_2] = Z_2$$

and the other brackets being zero. The completely solvable Lie group G^{BG} corresponding to $\mathfrak{g}^{\mathrm{BG}}$ can be expressed as follows:

$$G^{\mathrm{BG}} = \left\{ \begin{pmatrix} e^{-t} & 0 & e^{-2t}x_1 & 0 & z_1 \\ 0 & e^{t} & 0 & e^{2t}x_2 & z_2 \\ 0 & 0 & e^{-2t} & 0 & y_1 \\ 0 & 0 & 0 & e^{2t} & y_2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix};\ t, x_i, y_i, z_i \in \mathbb{R},\ i = 1, 2 \right\}.$$

A solvable Lie algebra $\mathfrak{g}$ is called completely solvable if for each $X \in \mathfrak{g}$, $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ has only real eigenvalues. Hattori [7] proved that the Chevalley–Eilenberg cohomology of the completely solvable Lie algebra $H^*(\mathfrak{g})$ is isomorphic to the de Rham cohomology of the solvmanifold $H^*_{DR}(G/\Gamma)$, where G is the simply-connected Lie group corresponding to $\mathfrak{g}$ and Γ is a lattice of G.

We say that a compact symplectic manifold (M^{2m}, ω) has the Hard Lefschetz property if the mapping $L^k : H^{m-k}_{DR}(M) \to H^{m+k}_{DR}(M)$, where $L^k[\alpha] = [\omega^k \wedge \alpha]$, is an isomorphism for each $k \leqslant m$. It is well known that a compact Kähler manifold has the Hard Lefschetz property and its minimal model is formal. In the case of compact nilmanifolds, if $L^{m-1} : H^1_{DR}(M) \to H^{2m-1}_{DR}(M)$ is an isomorphism, or if its minimal model is formal, then M is a torus [1,6].

Note that the Lie group $G^{\mathrm{BG}} \times \mathbb{R}$ has a left invariant symplectic structure. It is known that if G^{BG} admits a lattice Γ, then the solvmanifold $G^{\mathrm{BG}}/\Gamma \times S^1$ does not have the Hard Lefschetz property [2]. In particular, $G^{\mathrm{BG}}/\Gamma \times S^1$ admits no Kähler structures. However, the minimal model of $G^{\mathrm{BG}}/\Gamma \times S^1$ is formal [3] (cf. [8]).

We consider a completely solvable Lie algebra $\mathfrak{g}^{(k_1,k_2)}$ given by

$$\mathfrak{g}^{(k_1,k_2)} = \mathrm{span}\{A, X_1, Y_1, Z_1, X_2, Y_2, Z_2\},$$
$$[X_1, Y_1] = Z_1, \qquad [X_2, Y_2] = Z_2,$$
$$[A, X_1] = k_1 X_1, \qquad [A, Y_1] = k_2 Y_1, \qquad [A, Z_1] = (k_1 + k_2) Z_1,$$
$$[A, X_2] = -k_1 X_2, \qquad [A, Y_2] = -k_2 Y_2, \qquad [A, Z_2] = -(k_1 + k_2) Z_2,$$

where $k_1, k_2 \in \mathbb{Z}$. Note that $\mathfrak{g}^{\mathrm{BG}} = \mathfrak{g}^{(1,-2)}$. The simply-connected completely solvable Lie group $G^{(k_1,k_2)}$ corresponding to $\mathfrak{g}^{(k_1,k_2)}$ can be expressed by

$$G^{(k_1,k_2)} = \left\{ \begin{pmatrix} e^{(k_1+k_2)t} & 0 & e^{k_2 t}x_1 & 0 & z_1 \\ 0 & e^{-(k_1+k_2)t} & 0 & e^{-k_2 t}x_2 & z_2 \\ 0 & 0 & e^{k_2 t} & 0 & y_1 \\ 0 & 0 & 0 & e^{-k_2 t} & y_2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix};\ t, x_i, y_i, z_i \in \mathbb{R},\ i = 1, 2 \right\}.$$

Now we state our main theorem.

Theorem 1. *The completely solvable Lie group $G^{(k_1,k_2)}$ admits a lattice.*

In Section 2, we generalize Theorem 1. Let $\mathfrak{n}$ be a nilpotent Lie algebra of $(m-1)$-dimension which admits a lattice. Then $\mathfrak{n}$ has a basis $\{X_1, \ldots, X_{m-1}\}$ whose the structure constants are integers.