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Extending equivariant maps into spaces with convex structure

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Abstract

We prove that if G is a compact Lie group, Y a G-space equipped with a topological local convex structure compatible with the action of G, then Y is a G-*ANE* for metrizable G-spaces. If, in addition, Y has a G-fixed point and admits a global convex structure compatible with the action of G, then Y is a G-*AE*. This is applied to show that certain hyperspaces related to the Banach–Mazur compacta are equivariant absolute extensors.

MSC: 54C55; 54B20; 54H15; 57S99

Keywords: G-*LCS* space; G-*ANE*; Hyperspace; Orbit space

1. Introduction

In the present paper we prove a general equivariant extension theorem for equivariant maps with values in G-spaces which possess a (local or global) topological convex structure compatible with the given action of a compact Lie group G. Our Theorems 3.2 and 3.3 extend essentially the existing versions of the equivariant Dugundji extension theorem (see [17,2–4]). At the same time these theorems are equivariant generalizations of some

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results in Himmelberg [16] and Curtis [14]. In Corollary 3.5 we extend these results to the case of proper actions of arbitrary (non-compact) Lie groups. Corollary 4.4 states that for a compact Lie group G, metrizable G-*ANE*'s are precisely those metrizable G-spaces that admit such a local G-convex structure. This characterization is especially useful when one considers hyperspaces of compact convex subsets of normed linear G-spaces. In this way we show that certain important spaces related to the famous Banach–Mazur compacta are equivariant absolute extensors. In conclusion, we also discuss some open questions.

2. Preliminaries

We refer to the monographs [13,22] for basic notions of the theory of G-spaces. However, below we recall some special definitions and facts that are necessary throughout the paper.

If G is a topological group and X is a G-space, for any $x \in X$ we denote the stabilizer (or the stationary subgroup) of x by $G_x = \{g \in G \mid gx = x\}$. For a subset $S \subset X$ and a subgroup $H \subset G$, $H(S)$ denotes the H-saturation of S, i.e., $H(S) = \{hs \mid h \in H,\ s \in S\}$. If $H(S) = S$, then we say that S is an H-invariant set. In particular, $G(x)$ denotes the G-orbit $\{gx \in X \mid g \in G\}$ of x. The orbit space is denoted by X/G. By G/H we will denote the G-space of cosets $\{gH \mid g \in G\}$ under the action induced by left translations.

A compatible metric ρ on a G-space X is called invariant or G-invariant, if $\rho(gx, gy) = \rho(x, y)$ for all $g \in G$ and $x, y \in X$.

We shall mean by a linear G-space, a real topological vector space L on which G acts continuously and linearly, i.e., $g(\lambda x + \mu y) = \lambda(gx) + \mu(gy)$ for every $g \in G$ and for all $x, y \in L$ and $\lambda, \mu \in \mathbb{R}$.

The terms "G-map" or "equivariant map" will include the continuity of the corresponding map.

A G-space Y is called an equivariant neighborhood extensor for a given G-space X (notation: $Y \in G\text{-}ANE(X)$), if for any closed invariant subset $A \subset X$ and any G-map $f : A \to Y$, there exist an invariant neighborhood U of A in X and a G-map $\psi : U \to Y$ that extends f. If, in addition, one can always take $U = X$, then we say that Y is an equivariant extensor for X (notation: $Y \in G\text{-}AE(X)$). The map ψ is called a G-extension of f.

If G is a compact group, then a G-space Y is called an equivariant absolute neighborhood extensor (notation: $Y \in G\text{-}ANE$), if $Y \in G\text{-}ANE(X)$ for any metrizable G-space X. Similarly, if $Y \in G\text{-}AE(X)$ for any metrizable G-space X, then Y is called an equivariant absolute extensor (notation: $Y \in G\text{-}AE$).

If G is the trivial group, we just get from here the definitions of the ordinary classes *ANE* and *AE*, respectively.

The notion of a slice is the key tool in our proofs; let us recall it:

Definition 2.1 [22]. Let G be a topological group, $H \subset G$ a closed subgroup and X a G-space. A subset $S \subset X$ is called an H-slice in X, if:

(1) S is H-invariant,
(2) the saturation $G(S)$ is open in X,