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On two-dimensional planar compacta not homotopically equivalent to any one-dimensional compactum

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Abstract

The paper provides examples of planar "homotopically two-dimensional" compacta, (i.e., of compact subsets of the plane that are not homotopy equivalent to any one-dimensional set) that have different additional properties than the first such constructed examples (amongst them cell-like, trivial π_1, and "everywhere" homotopically two-dimensional). It also points out that open subsets of the plane are never homotopically two-dimensional and that some homotopically two-dimensional sets cannot be in such a way decomposed into homotopically at most one-dimensional sets that the Mayer–Vietoris Theorem could be straightforwardly applied.

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1. Introduction

The well-known example of Barratt and Milnor [1] shows that there exists a Peano continuum X in $\mathbb{R}^3$ whose singular homology groups with integer coefficients $H_n(X)$ are nontrivial for every $n > 1$.

Planar sets behave more naturally with respect to their algebraic topology: By [13,6] they are all aspherical, and by [14] also acyclic *in higher dimensions* with respect to singular homology, i.e., $H_n(M)$ of any planar set M, $M \subset \mathbb{R}^2$, is trivial for all $n > 1$. The proof in [14] is very delicate and complicated. However, for certain kinds of spaces there exist simpler proofs, e.g., for the spaces which are homotopy equivalent to a 1-dimensional separable metric space [7]. Not all planar sets are homotopy equivalent to 1-dimensional spaces: Most recently Cannon and Conner [5] characterized when a discrete subset of the plane satisfies that its complement has this property, but the first (not codiscrete) examples with this property have already appeared before ([6, §5], [13, A.4.13]). The fundamental group of these spaces are uncountable.

This is one of the aspects in which we improve upon these results by providing the following three examples:

Brief description of our main examples

All these spaces are planar continua that are not homotopy equivalent to any one-dimensional space. In addition

Example 1 (see Fig. 1(a)) *shows that it is not necessary to have fundamental group to obtain this effect*. Indeed this space, which is constructed by wedging a null-sequence of two comb-spaces to the boundary of a disk, is path-connected, simply connected and cellular (cf. 3.1–3.2).

Example 2 (see Fig. 1(b)) *shows that this effect can also be achieved for a Peano continuum that is a disjoint union of a disk and open intervals*. It is built by similarly wedging countably many Hawaiian Earrings to the boundary of a disk (cf. 3.1–3.2).

Example 3 (see Fig. 1(c)) *shows that one can also construct planar Peano continua* "*no part*" *of which is homotopy equivalent to any one-dimensional set,* more precisely the continuum is everywhere homotopically 2-dimensional (cf. Definition 2.1(ii)). Here the construction is based on the Sierpiński Carpet, by filling (instead of removing) an appropriate subset of all holes (cf. 3.3–3.4).

More precise definitions of these examples and proofs of these claims will follow in Section 3. Section 4 will then be devoted to proving the following facts:

Proposition 1.1. *A subset of the plane that is not homotopy equivalent to a one-dimensional space can neither be*

(i) *simply connected and a Peano continuum* (*cf. Proposition* 4.1), *nor be*
(ii) *open* (*cf. Remark* 4.2).