Available online at www.sciencedirect.com
SCIENCE DIRECT®
Topology and its Applications 152 (2005) 219–225
Topology and its Applications
www.elsevier.com/locate/topol

Multiplications in solenoids as hyperbolic attractors

Floris Takens

Grotestraat 31, Bedum 9781 HB, Netherlands

Dedicated to Jan M. Aarts on the occasion of his retirement

Abstract

The solenoid was first introduced by Vietoris, motivated by questions from algebraic topology. It later appeared in the study of dynamical systems. This paper discusses the history of solenoids and settles an isomorphism problem.

MSC: 37D05; 37D45

Keywords: Solenoid; Hyperbolic attractor; Cantor groups

1. Introduction

The solenoid has an interesting history in geometry and dynamics.

It was first introduced by Vietoris in [9] as part of the construction of an example of a continuum for which the fundamental group, in the sense of Vietoris, depends on the base point. We remind here that in this paper Vietoris introduced a special type of homology, cohomology, and fundamental groups for compact metric spaces. This construction was generalized and modified by various authors, see [3] for an account of the history. The resulting homology and cohomology groups are now named after Čech; the fundamental groups have been forgotten. In the notation of our Section 3 Vietoris constructed in this paper the solenoid Σ_2. In the following, if we speak of *the* solenoid, we mean Σ_2.

E-mail address: f.takens@math.rug.nl (F. Takens).

0166-8641/$ – see front matter
doi:10.1016/j.topol.2004.10.010

Independently and around the same time Van Danzig made an extensive study of these solenoids [2] as a class of spaces with very strong homogeneity properties without being manifolds. This homogeneity is due to the fact that these spaces can be given the structure of a topological group. In our notation, Van Danzig restricted himself in this publication to the solenoids $\boldsymbol{\Sigma_n}$, for $n \in \mathbf{N}$ (in the notation of our Section 3). One of his results was a topological classification of these spaces. The topological classification of the more general solenoids, was completed by McCord [5] and simplified by Aarts and Fokkink [1].

In dynamics it first appeared as an example, due to Nemytskii, of the closure of an almost periodic motion which is not quasiperiodic. This example can be found in [6, Chapter V, Section 8] which is a translation of Russian publications in the period 1947–1952. The relation between the group structure and the dynamical structure is the following. The solenoid contains a dense subgroup which is isomorphic to the additive group of the reals $\mathbf{R}$ (and, as a subspace, is an immersion of $\mathbf{R}$). If we identify the reals with this subgroup, then the time evolution over an interval t corresponds to the translation $x \mapsto x + t$. As far as I know, it is not known which solenoids are conjugated to the closure of an orbit of a *smooth* dynamical system. We recall that the closure of a quasiperiodic motion is not only a topological group but even a torus group.

Later the solenoid was used by Williams, first as an example of a space on which an 'unstable' homeomorphism could be defined (this is what one calls nowadays an expansive homeomorphism) in [10] and later as one of the first examples of a hyperbolic, chaotic, and fractal attractor in [11]. We describe this example in detail in our Section 2.

It turns out that the dynamics in $\boldsymbol{\Sigma_2}$, introduced by Williams, can be easily described in terms of the group structure. It is just multiplication by 2, i.e., the map sending x to $x + x$. In this paper we give a classification of those multiplications in general solenoids which are conjugated to a hyperbolic attractor.

2. The solenoid as attractor of a diffeomorphism

We consider a solid torus $T = S^1 \times D^2$ in dimension 3 and a diffeomorphism of T to a subset of T such that in the S^1 direction we have an expanding map of degree 2 and in the D^2 direction we have a (strong) contraction. Taking $s \in \mathbf{R}$ mod 1 as a coordinate on S^1 and x, y, with $x^2 + y^2 \leqslant 1$, as coordinates on D^2, an example of such a map is given by:

$$\varphi(s, x, y) = \bigl(2s \bmod 1, c_1 \cos(2\pi s) + c_2 x, c_1 \sin(2\pi s) + c_2 y\bigr),$$

with $0 < c_2 < c_1$ and $c_1 + c_2 < 1$. The *solenoid* is then defined as $S = \bigcap_i \varphi^i(T)$. The dynamics on S is given by $\varphi|_S$.

This diffeomorphism is (*structurally*) *stable* in the sense that if $\tilde{\varphi} : T \to T$ is C^1 sufficiently close to φ, then there is a homeomorphism $h : T \to T$ such that $h\varphi = \tilde{\varphi}h$. This stability is a consequence of the fact that φ is *hyperbolic* in the sense that for each $q \in S$ there is a splitting $T_q = E^s(q) + E^u(q)$ of the 3-dimensional space of tangent vectors at q such that, under repeated application of φ, vectors in $E^s(q)$, respectively $E^u(q)$, are exponentially contracted, respectively expanded; in fact the contracting vectors are tangent to the discs $\{s\} \times D^2$. For the theory of hyperbolic dynamical systems and their structural stability see, e.g., [8].