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Zero-selectors and GO spaces ☆

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Dedicated to Professor Jan Aarts

Abstract

We are dealing with Vietoris continuous zero-selectors, i.e., they choose for each non-empty closed set F an isolated point in F. We show that the presence of a continuous zero-selector even on a small class of non-empty closed sets of a space X implies that X is scattered if X is metrizable or non-Archimedean or a P-space. Finally, using continuous zero-selectors, we characterize suborderable spaces which are subspaces of ordinals.

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0. Introduction

We continue the study of continuous *zero-selectors*, i.e., selectors, which are continuous with respect to the Vietoris topology on the family of non-empty closed sets of a given space X and which choose a (relatively) isolated point from each non-empty closed set. Clearly, the existence of an arbitrary zero-selector for X implies that X must be scattered.

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In the part 1.1, we are going to show that, under some additional restrictions to the space, X must be scattered even when a continuous zero-selector acts on a small subfamily of closed sets (see Theorem 1.5 below and its corollaries). In the part 1.2, following reasoning from [15], we show that the density of a regular space X with a continuous zero-selector is equal to the cardinality of X.

Continuous zero-selectors can be defined quite easily for ordinals: just take for each non-empty closed set its minimum. In Section 2, we are dealing with the "opposite" problem: which suborderable spaces with a continuous zero-selector are homeomorphic to a subspace of ordinals? We give a characterization of these suborderable spaces in Theorem 2.9 and we present examples that conditions used in Theorem 2.9 cannot be weakened. Let us recall in this connection that continuous zero-selectors were used to characterize compact ordinal spaces in [9], i.e., it was shown in particular that any compact space with a continuous zero-selector is homeomorphic to a space of ordinals. This fact was afterwards generalized for pseudocompact spaces [2].

1. Zero-selectors

All spaces considered in this paper are Hausdorff. Let $\mathfrak{F}(X)$ be the set of all non-empty closed subsets of X, equipped with the Vietoris topology [16,7]. A base for the Vietoris topology on a subspace $\mathfrak{A}$ of $\mathfrak{F}(X)$ consists of all sets of the form:

$$\langle U^0, U^1, \ldots, U^n \rangle = \left\{ F \in \mathfrak{A}\colon\ F \subseteq \bigcup_{i \leqslant n} U^i \text{ and } F \cap U^i \neq \emptyset \text{ for every } i \leqslant n \right\}$$

where $U^0, \ldots, U^n$ are open subsets of X.

A (continuous) selector on a subspace $\mathfrak{A}$ of $\mathfrak{F}(X)$ is a (continuous) map $\sigma : \mathfrak{A} \to X$ such that $\sigma(F) \in F$ for every $F \in \mathfrak{A}$, The selector σ is said to be a zero-selector provided that $\sigma(F)$ is relatively isolated in F for each $F \in \mathfrak{A}$. We say that X has a continuous (zero-)selector if there exists a continuous (zero-)selector on the whole $\mathfrak{F}(X)$.

The set of cluster points of a set E is denoted by E'.

1.1. Zero-selectors on smaller families

We start with some facts which will be useful in the sequel.

Lemma 1.1. *Let σ be a continuous selector on a subspace $\mathfrak{A}$ of $\mathfrak{F}(X)$, $C \in \mathfrak{A}$, $p = \sigma(C)$. Assume $p \in C'$. Then for every neighbourhood W of p there exists a non-empty finite subset $\Gamma(W)$ of $C \setminus \{p\}$ such that $\sigma(G) \in W$ for each $G \in \mathfrak{A}$, with $\Gamma(W) \subseteq G \subseteq C$.*

Proof. By the continuity of σ, there exist non-empty open sets $U^0, U^1, \ldots, U^n$ such that $C \in \langle U^0, U^1, \ldots, U^n \rangle \subseteq \sigma^{-1}(W)$. Since $p \in C'$, we can choose a point $p_i \in U^i \cap C$, for every $i \leqslant n$. The required set is $\Gamma(W) = \{p_1, \ldots, p_n\}$. □

Notice that the condition $p \in C'$ in the above lemma was needed to ensure that $\Gamma(W)$ does not contain p. If p is isolated in C then it is not excluded that $U^i \cap C = \{p\}$ for some i.