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Topology and its Applications 153 (2005) 747–755
Topology and its Applications
www.elsevier.com/locate/topol

Productivity of Zariski-compactness for constructs of affine spaces

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Received 21 November 2003; received in revised form 15 June 2004

Abstract

We study compactness for hereditary coreflective subconstructs **X** of S**SET**, the construct of affine spaces over the two point set S and with affine maps as morphisms, endowed with the Zariski closure operator z. We formulate necessary conditions for productivity of z-compactness. Moreover, if in **X** arbitrary products of quotients are quotients, then our conditions are also sufficient. We apply the results to some well-known subconstructs of S**SET**, in particular we investigate situations in which another sufficient condition for productivity of compactness, known as finite structure property for products (FSPP), is not fulfilled by the Zariski closure.

MSC: 54B30; 54A05; 54D30

Keywords: Hereditary coreflective subconstruct; Structured set; Zariski-compactness; Finite structure property; Zariski closure

1. Introduction

An affine space X over the two point set $S = \{0, 1\}$ is a structured set, where the structure on the underlying set X is a collection of subsets of X. The sets belonging to the structure are called the "open" sets of X. An affine map from $\mathsf{X} \to \mathsf{Y}$ is a function f such

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0166-8641/$ – see front matter
doi:10.1016/j.topol.2005.01.007

that inverse images of open sets are open. An affine space can of course be equivalently described by defining a collection of maps to S instead of giving a family of open subsets of X. It was pointed out by Giuli in [10] that affine spaces and maps coincide with normal Boolean Chu spaces and continuous maps, as introduced by Pratt for modelling concurrent computation [16]. More general settings have been considered where S is replaced by some arbitrary fixed set K [8,11].

In this paper we will restrict ourselves to affine spaces X for which $\emptyset$ and X are open. As in [10], the corresponding construct of affine spaces and affine maps as morphisms is denoted by S**SET**. The space $\mathsf{S} = (S, \{\emptyset, X, \{1\}\})$ is called the Sierpinski affine space. The construct S**SET** is well-fibered topological and it is endowed with the usual $(\mathcal{E}, \mathcal{M})$ factorization structure, where $\mathcal{E}$ is the class of epimorphisms and where $\mathcal{M}$ is the class of embeddings. On this category lives a natural closure operator, called the Zariski closure z. For $\mathsf{M} \to \mathsf{X}$ in $\mathcal{M}$ and $x \in X$ we put $x \in z(\mathsf{M})$ if and only if for every α and β, affine maps to S, whenever α and β coincide on M they also coincide in x. The Zariski closure is the regular closure associated with the class of T_0-objects in S**SET** in the sense of [15]. Moreover this closure operator is known to be hereditary. Well-known constructs such as **TOP**, the construct of topological spaces or **CL**, the construct of closure spaces are hereditary coreflective full subconstructs of S**SET**. Such coreflective and hereditary subconstructs inherit the factorization structure as well as the Zariski closure operator from S**SET**. A thorough investigation on categorical completeness (absolutely z-closedness) in $\mathbf{T_0X}$, the class of T_0-objects for some hereditary and coreflective subconstruct **X** of S**SET**, has been carried out by Giuli in [10] and we make frequent use of the results obtained there. In [9] internal characterizations are given for z-completeness in $\mathbf{T_0X}$.

In this paper we will be dealing with the categorical notion of compactness as developed in [6] and applied to the Zariski closure z on S**SET** and on some of its subconstructs. In particular we will investigate productivity of Zariski-compactness. For an arbitrary hereditary coreflective subconstruct **X**, we prove that productivity of z-compactness in **X** implies the property that either all z-compact objects are indiscrete or z-compactness coincides with z-completeness (i.e., absolutely z-closedness) on T_0-objects of **X**. Moreover we prove that, when in **X** arbitrary products of quotients are quotients, our conditions are also sufficient. We apply these results to some well-known subconstructs such as **CL** and **TOP** and to some of their subconstructs, in particular to some constructs on which the Zariski closure fails to satisfy the FSPP condition (the finite structure property for products) [6].

We thank the referee for the many valuable suggestions which improved the paper a lot, in particular for questioning the setting that we originally considered for our main Theorem 3.4.

2. Notations and preliminary results

The Sierpinski space S plays an important role in S**SET**, in particular it is an initially dense object. Also S**SET** has some other nice properties. The following preliminary result about the interaction of products and quotients in S**SET** will be usefull for us.

Proposition 2.1. *In* S**SET** *arbitrary products of quotients are quotients.*