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Partial metric monoids and semivaluation spaces

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Abstract

Stable partial metric spaces form a fundamental concept in Quantitative Domain Theory. Indeed, *all* domains have been shown to be quantifiable via a stable partial metric.

Monoid operations arise naturally in a quantitative context and hence play a crucial role in several applications. Here, we show that the structure of a stable partial metric monoid provides a suitable framework for a unified approach to some interesting examples of monoids that appear in Theoretical Computer Science. We also introduce the notion of a semivaluation monoid and show that there is a bijection between stable partial metric monoids and semivaluation monoids.

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1. Introduction

Throughout this paper the letters $\mathbb{R}, \mathbb{R}^+$ and ω will denote the set of real numbers, of nonnegative real numbers and of nonnegative integer numbers, respectively.

Matthews introduced in [14] the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks, and obtained, among other results, a nice relationship between partial metric spaces and the so-called weightable quasi-metric spaces. These structures have been applied to obtain an extensional treatment of lazy data flow deadlock in [15].

Let us recall that a *partial metric* on a (nonempty) set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(i) $x = y \Longleftrightarrow p(x,x) = p(x,y) = p(y,y)$;
(ii) $p(x,x) \leqslant p(x,y)$;
(iii) $p(x,y) = p(y,x)$;
(iv) $p(x,z) \leqslant p(x,y) + p(y,z) - p(y,y)$.

A *partial metric space* is a pair (X, p) such that X is a (nonempty) set and p is a partial metric on X.

Each partial metric p on X generates a T_0-topology $\mathcal{T}(p)$ on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon)\colon\ x \in X,\ \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X\colon\ p(x,y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Note that contrarily to the metric case, some open p-ball may be empty [14, p. 187].

The following is a simple but useful example of a partial metric space.

For each pair $x, y \in \mathbb{R}^+$ let $p(x,y) = x \vee y$. Then p is a partial metric on $\mathbb{R}^+$ and thus $(\mathbb{R}^+, p)$ is a partial metric space.

If (X, p) is a partial metric space, then $(X, \sqsubseteq_p)$ is clearly an ordered set, where $x \sqsubseteq_p y \Longleftrightarrow p(x,x) = p(x,y)$.

In the sequel $\sqsubseteq_p$ will be called the *associated* (*partial*) *order* of p.

In [23] it is shown that *all* domains are quantifiable via a partial metric induced by a suitable semivaluation.

We focus in the following on three well-known Computer Science examples of monoids for which the notion of a (stable) partial metric monoid will provide a unifying concept.

The interval domain (or the partial real line) forms a model for a programming language for higher-order exact real number computation [5]. It consists of the set $I(\mathbb{R})$ of all nonempty closed and bounded real intervals ordered by reverse inclusion, together with an artificial least element $\perp$. In [14] (see also [8,16]) a partial metric p is defined on $I(\mathbb{R})$ such that its associated order coincides with the reverse inclusion order and thus $(I(\mathbb{R}), \sqsubseteq_p)$ is a meet semilattice as it is observed in [22]. We shall denote by $I([0,1])$ the set of all nonempty closed and bounded intervals contained in $[0,1]$. It was proved in [6] that $I([0,1])$ can be equipped with a suitable structure of monoid for which $[0,1]$ is the neutral element. For simplicity and without essential loss of generality, we shall refer in the sequel to $I([0,1])$ as the interval domain.

If Σ^∞ denotes the set of all finite and infinite “words” over a nonempty alphabet Σ, then $(\Sigma^\infty, \sqsubseteq)$ is a meet semilattice where $\sqsubseteq$ is the prefix order on Σ^∞. Furthermore