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Countable sets, BCO spaces and selections ☆

Peng-Fei Yan [a,1] Shou-Li Jiang [b,*]

[a] Department of Mathematics, Nanjing Normal University, Nanjing 210097, China
[b] Department of Mathematics, Shandong University, Jinan 250100, China

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Abstract

The main purpose of this paper is to give the selection theorems in BCO spaces which unify and generalize some known results. Also, the relations between countable sets and selections are discussed.

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1. Introduction

Let X and Y be topological spaces, and 2^Y stand for the family of non-empty subsets of Y. We write

$$F(Y)=\{S\in 2^Y:\ S \text{ is closed}\},$$
$$C(Y)=\{S\in F(Y):\ S \text{ is compact}\},$$
$$K(Y)=\{S\in F(Y):\ F \text{ is finite}\}.$$

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* Corresponding author.
E-mail addresses: ypengfei@sina.com (P.-F. Yan), jiang@math.sdu.edu.cn (S.-L. Jiang).
1 Current address: Department of Mathematics, Anhui University, Hefei 230039, China

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For a metric space (Y, d), $\mathcal{F}(Y) = \{S \in 2^Y\colon\ S$ is a complete subset of $Y\}$.

A set-valued mapping $\Phi : X \to 2^Y$ is lower semi-continuous (upper semi-continuous) or l.s.c. (u.s.c.), if the set

$$\Phi^{-1}(U) = \{x \in X\colon\ \Phi(x) \cap U \neq \emptyset\}$$

is open (respectively, closed) in X for every open (respectively, closed) U of X.

A function $f : X \to Y$ is a continuous selection of Φ if f is continuous and $f(x) \in \Phi(x)$ for each $x \in X$.

For $l.s.c.$ mappings, Michael obtained the following theorems:

Theorem 1.1 [5]. *Let X be a zero-dimensional paracompact space, (Y, d) a metric space, and an $l.s.c.$ mapping $\Phi : X \to \mathcal{F}(Y)$, then there exists a continuous selection of Φ.*

Theorem 1.2 [6]. *Let X be a paracompact space, (Y, d) a metric space, and an $l.s.c.$ mapping $\Phi : X \to \mathcal{F}(Y)$, then there exists an $l.s.c.$ mapping $\varphi : X \to C(Y)$ and an $u.s.c.$ mapping $\phi : X \to C(Y)$ such that $\varphi \subset \phi \subset \Phi$.*

Theorem 1.3 [7]. *Let X be a regular countable space, Y a first countable space, then every $l.s.c$ mapping $\Phi : X \to 2^Y$ has a continuous selection.*

The basic methods to construct selections in Theorems 1.1 and 1.2 is to find a sequence of functions to approx Φ, hence the metribility of Y is important. A natural question is that the metric structure of image spaces is necessary or not? Motivated by Theorem 1.3, the first author [9] found the key role of BCO in constructing selections. Recently, Alleche and Calbrix [1] also given some relations between BCO and selections. The purpose of this paper is to strength these results, show that in many cases, BCO structures can replace metric to construct selections, and give a direct proof of a selection theorem in BCO spaces. Also, we prove that the countableness of X is necessary in Theorem 1.3.

Let us recall the concept of BCO.

A base $\mathcal{B}$ for a space X is called a base of countable order or BCO if for every $x \in X$ and every strictly decreasing sequence (B_n) of elements of $\mathcal{B}$ containing x, (B_n) is a base of x. Wick and Worrel [8] obtained the following properties of BCO.

Lemma 1.4. *Let $\mathcal{B}$ be a countable order base of X, then there exists a sequence $(\mathcal{B}_n)$ of bases of X consisted of subsets of $\mathcal{B}$ satisfying*

* *For each $x \in X$, if $x \in B_n \in \mathcal{B}_n$ and $B_{n+1} \subseteq B_n$ for every $n \in N$, then (B_n) is a base of x.*

In [1], Alleche and Calbrix introduce the notion of monotonically completness on a subset. We say that a base $\mathcal{B}$ for a space X is monotonically complete on a subset A of X, if for every decreasing sequence (B_n) of elements of $\mathcal{B}$ such that $B_n \cap A \neq \emptyset$ for every $n \in N$, then $\bigcap_{n\in N} \overline{B_n} \neq \emptyset$. If $A = X$, we call that $\mathcal{B}$ is monotonically complete. $\mathcal{B}$ is monotonically complete on $\mathcal{F}$, if $\mathcal{B}$ is monotonically complete on every element of $\mathcal{F}$.

Define $\mathcal{F}_{\mathcal{B}}(X) = \{F\colon\ F \in F(X), \mathcal{B}$ is monotonically complete on $F\}$.