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Embeddability properties of countable metric spaces

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Abstract

For subspaces X and Y of $\mathbb{Q}$ the notation $X \leqslant_h Y$ means that X is homeomorphic to a subspace of Y and $X \sim Y$ means $X \leqslant_h Y \leqslant_h X$. The resulting set $\mathcal{P}(\mathbb{Q})/\sim$ of equivalence classes $\overline{X} = \{Y \subseteq \mathbb{Q}\colon\ Y \sim X\}$ is partially-ordered by the relation $\overline{X} \leqslant_h \overline{Y}$ if $X \leqslant_h Y$. It is shown that $(\mathcal{P}(\mathbb{Q}), \leqslant_h)$ is partially well-ordered in the sense that it lacks infinite anti-chains and infinite strictly descending chains. A characterization of $(\mathcal{P}(\mathbb{Q}), \leqslant_h)$ in terms of scattered subspaces of $\mathbb{Q}$ with finite Cantor–Bendixson rank is given and several results relating Cantor–Bendixson rank to this embeddability ordering are established. These results are obtained by studying a local homeomorphism invariant (*type*) for countable scattered metric spaces.

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1. Introduction

This paper contains a detailed study of the embeddability properties of countable metric spaces. The only interesting case is when these spaces are scattered, and under the assumption of local compactness the theory is quite simple. The central technical theorem presented here reduces the investigation to studying countable scattered metric spaces of finite Cantor–Bendixson rank. For these spaces, a theory of *types* is developed which cap-

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tures the local embeddability properties of points. The main results then follow from some simple set-theoretic lemmas.

The symbol $\bigoplus_I X_i$ denotes the topological sum (disjoint union) of the spaces X_i $(i \in I)$ and $\prod_I X_i$ is their product. The Greek letters α, β, λ denote ordinal numbers (each is the set of smaller ordinals), ω is the first infinite ordinal, ω_1 is the first uncountable ordinal and $\mathbb{N} = \{1, 2, \ldots\}$ denotes the countably infinite discrete space. We use $d(x, y)$ to denote the distance from x to y in a metric space, and we extend this as usual to define $d(A, B)$ for subsets A and B; $D(A) := \sup_{x,y\in A} d(x, y)$ denotes the diameter of A. The logical symbols $\wedge$ (and) and $\vee$ (or) are used on occasion. Throughout the paper, let

$$M_0 := \{1/n\colon\ n \in \mathbb{N}\} \cup \{0\},$$
$$M_1 := \left\{1/n + 1/n^2m\colon\ m, n \in \mathbb{N}\right\} \cup \{0\},$$
$$I_n := \left(\frac{2^{2n-2}-1}{2^{2n-2}}, \frac{2^{2n-1}-1}{2^{2n-1}}\right) \cap \mathbb{Q} \quad (n \in \mathbb{N}).$$

That is, M_0 is a convergent sequence (with its limit point), M_1 is a "convergent sequence" of convergent sequences without their limit points, and I_n $(n \in \mathbb{N})$ is a sequence of pairwise completely separated open intervals so that for all $\varepsilon > 0$, only finitely many of these intervals are not contained in $(1 - \varepsilon, 1)$.

For a topological space X and an ordinal number α, the αth *Cantor–Bendixson derivative* (denoted $X^{(\alpha)}$) is defined inductively by letting $X^{(0)} := X$, $X^{(\lambda+1)} := \{x \in X^{(\lambda)}\colon\ x$ is not isolated in $X^{(\lambda)}\}$ for successor ordinals $\lambda + 1$, and $X^{(\alpha)} := \bigcap_{\beta<\alpha} X^{(\beta)}$ for limit ordinals α. A point $x \in X^{(\alpha)} \setminus X^{(\alpha+1)}$ will be called α-isolated. Note that if U is a neighborhood of an α-isolated point x in a space X then x will also be an α-isolated point in the space U. Furthermore, $X^{(\alpha)}$ is a closed subspace of $X^{(\beta)}$ for all $\beta \leqslant \alpha$.

A space X is *scattered* if every non-empty subspace of X has an isolated point. Notice that M_1 is a countable scattered metric space that is not locally compact. It is perhaps interesting to note that any non-locally compact countable metric space with precisely one non-isolated point is homeomorphic to M_1. It is not hard to see that if X is scattered and countable, then there is a countable ordinal β such that $X^{(\beta)} = \emptyset$. The smallest such ordinal is denoted $N(X)$ and is called the (Cantor–Bendixson) *rank* of X.

For topological spaces X and Y, write $X \leqslant_h Y$ if X embeds homeomorphically in Y (i.e., if X is homeomorphic to a subspace of Y). For any family of spaces $\mathcal{K}$, the relation $X \sim Y$ if and only if $X \leqslant_h Y \leqslant_h X$ is an equivalence relation on $\mathcal{K}$. Let $\overline{X}$ denote the equivalence class of X under this equivalence relation. Ordering these equivalence classes by making $\overline{X} \leqslant_h \overline{Y}$ whenever $X \leqslant_h Y$ for some (equiv. all) $X \in \overline{X}, Y \in \overline{Y}$ makes $(\mathcal{K}/{\sim}, \leqslant_h)$ a partially-ordered set. The case $\mathcal{K} = \mathcal{P}(\mathbb{R})$ has been investigated extensively by McMaster, McCluskey, and Watson in [7] and [6], though no complete characterization of $(\mathcal{P}(\mathbb{R})/{\sim}, \leqslant_h)$ is presently known to the author. The case $\mathcal{K} = \mathcal{P}(\mathbb{Q})$ is studied here.

The following classical result of Sierpinski is used throughout without further mention (for a proof, see 6.2.A(d) of [1] or [8]).

Theorem 1. *If X is a countable metric space without isolated points, then X is homeomorphic to $\mathbb{Q}$.*