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On Σ^*-spaces and strong Σ^*-spaces of countable pseudocharacter

Liang-Xue Peng

College of Applied Science, Beijing University of Technology, Beijing 100022, China

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Abstract

Lin raised the following open problem in 1995: If X is a strong Σ^*-space and every point of X is a G_δ-set of X, is X a strong Σ-space? In this paper, this problem is answered. The main conclusion is that if X is a strong Σ^*-space and every point of X is a G_δ-set of X, then X is a strong Σ-space. As a corollary, we have that a T_2-space is a σ-space iff it is a Σ^*-space with a G_δ-diagonal. Some conditions that imply Σ^*-property are discussed.

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Introduction

The properties of Σ-spaces, strong Σ-spaces, Σ^*-spaces and strong Σ^*-spaces have been studied by many topologists. Now, let us review these definitions. If $\mathcal{K}$ is a cover of a space X, then a cover $\mathcal{P}$ is called a $(\text{mod}\,\mathcal{K})$-network for X if, whenever $K \subset U$ with $K \in \mathcal{K}$ and U open in X, then $K \subset P \subset U$ for some $P \in \mathcal{P}$. A space X is a Σ-space (Σ^*-space) if it has a σ-locally finite (σ-hereditarily closure-preserving) closed $(\text{mod}\,\mathcal{K})$-network for some cover $\mathcal{K}$ of X by countably compact sets. In the definitions, if $\mathcal{K}$ is a cover of X by compact sets, then X is called strong Σ-space (strong Σ^*-space) (cf. [1–4]). We know that a collection $\mathcal{P} = \{P_\lambda\colon \lambda \in \Lambda\}$ is hereditarily closure-preserving (abbreviated HCP) if for any collection $\{B_\lambda\colon \lambda \in \Lambda\}$ with $B_\lambda \subset P_\lambda$ is closure-preserving.

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From these definitions we know that every strong Σ-space is a Σ-space, and every strong Σ^*-space is a Σ^*-space. In [2], Okugyama proved if X is a Σ^*-space for which every open set is an F_σ-set, then X is a Σ-space. In [3], it is proved that every T_2 strong Σ-space with a G_δ-diagonal is a σ-space. Thus Lin raised the following problem in [4] in 1995.

Problem [4]. If X is a strong Σ^*-space and every point of X is a G_δ-set of X, is X a strong Σ-space?

In this paper we give a positive answer to Lin's problem. This, together with Theorem 3.29 of the paper [3] implies that a Hausdorff space X is a σ-space if and only if it is a Σ^*-space with a G_δ-diagonal. We also discuss some conditions which imply Σ^*-property.

All spaces of the paper are T_1-spaces, and all maps are continuous. Let N be the set of all natural numbers.

1. Some conditions which imply Σ^*-property

Lemma 1.1 [2]. *X is a Σ-space iff X has a sequence $\{\mathcal{P}_n\colon n \in N\}$ of locally finite closed covers of X such that any sequence $\{x_n\colon n \in N\}$ with $x_n \in C(x, \mathcal{P}_n)$ for some $x \in X$ has a cluster point* (*where $C(x, \mathcal{P}_n) = \bigcap\{P\colon x \in P,\ P \in \mathcal{P}_n\}$*).

Lemma 1.2 [2]. *If X is a Σ^*-space, then X has a sequence $\{\mathcal{P}_n\colon n \in N\}$ of HCP closed covers of X, such that any sequence $\{x_n\colon n \in N\}$ with $x_n \in C(x, \mathcal{P}_n)$ for some $x \in X$ has a cluster point.*

Lemma 1.3 [8]. *If $\mathcal{P}$ is a HCP family of X, then $\{P_1 \cap P_2 \cap \cdots \cap P_n\colon P_i \in \mathcal{P},\ i \leqslant n\}$ is also a HCP family of X, $n \in N$.*

For the sake of convenience, let us denote some items.

The condition (I): X has a sequence $\{\mathcal{P}_n\colon n \in N\}$ of HCP closed covers of X such that any sequence $\{x_n\colon n \in N\}$ with $x_n \in C(x, \mathcal{P}_n)$ for some $x \in X$ has a cluster point.

If $\mathcal{P} = \bigcup\{\mathcal{P}_n\colon n \in N\}$, $\mathcal{P}_n$ is a family of subsets of X, $n \in N$. Let us denote $D_n = \{x\colon x \in X,\ \mathcal{P}_n$ is not point finite at $x\}$, $G_m = \{x\colon x \in X,\ |\bigcap\{P\colon x \in P,\ P \in \mathcal{P}_m\}| < \omega\}$, $C(x, \mathcal{P}_n) = \bigcap\{P\colon x \in P,\ P \in \mathcal{P}_n\}$ and $C(x, \mathcal{P}) = \bigcap\{C(x, \mathcal{P}_n)\colon n \in N\}$.

One natural question is that whether the converse of Lemma 1.2 is true. Now let us discuss some properties of spaces which satisfy the condition (I). If X is a Σ^*-space, let $\mathcal{P} = \bigcup\{\mathcal{P}_n\colon n \in N\}$ be the σ-HCP closed $(\mathrm{mod}\,\mathcal{K})$-network for some cover $\mathcal{K}$ of X by countably compact sets, then $D_n \subset G = \bigcup\{G_m\colon m \in N\}$ for each $n \in N$, and $D = \bigcup\{D_n\colon n \in N\}$ is a σ-discrete subset of X if and only if $G = \{G_m\colon m \in N\}$ is a σ-discrete subset of X (cf. [7]). Now, let us show that these conclusions are also true if X satisfies the condition (I).

Lemma 1.4. *If X satisfies the condition* (I), *then $D_n \subset G = \bigcup\{G_m\colon m \in N\}$ for each $n \in N$.*