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Paracompactness and the Lindelöf property in countable products

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Abstract

In this paper, we prove the following: Let Y be a perfect paracompact (hereditarily Lindelöf) space and $\{X_n\colon n \in \omega\}$ be a countable collection of Čech-scattered paracompact (Lindelöf) spaces, then the product $Y \times \prod_{n\in\omega} X_n$ is paracompact (Lindelöf).

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1. Introduction

Since the notion of C-scattered spaces was introduced by Telgársky [12], C-scattered spaces play the fundamental role in the study of paracompactness (Lindelöf property) of products. A space X is said to be *scattered* if every nonempty subset A has an isolated point in A, and X is said to be *C-scattered* if for every nonempty closed subset A of X, there is a point $x \in A$ which has a compact neighborhood in A. Then scattered spaces and locally compact spaces are C-scattered. R. Telgársky proved the following:

(A) (Telgársky [12]) If X is a C-scattered paracompact (Lindelöf) space, then $X \times Y$ is paracompact (Lindelöf) for every paracompact (Lindelöf) space Y.

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Let $\mathcal{L}$ be the class of all spaces whose product with every hereditarily Lindelöf space is Lindelöf. E. Michael asked whether $\mathcal{L}$ is closed with respect to countable products. Alster [2,3] gave a negative answer to this problem and proved

(B) (Alster [1]) If $\{X_n: n \in \omega\}$ is a countable collection of C-scattered Lindelöf spaces, then the product $\prod_{n\in\omega} X_n \in \mathcal{L}$.

For paracompactness of countable products, we have

(C) (Alster [4]) If Y is a perfect paracompact space and $\{X_n: n \in \omega\}$ is a countable collection of scattered paracompact spaces, then the product $Y \times \prod_{n\in\omega} X_n$ is paracompact.

(D) (Friedler et al. [7], Hohti and Pelant [8]) If $\{X_n: n \in \omega\}$ is a countable collection of C-scattered paracompact spaces, then the product $\prod_{n\in\omega} X_n$ is paracompact.

Hohti and Ziqiu [9] introduced the notion of Čech-scattered spaces, which is a generalization of C-scattered spaces. A space X is said to be Čech-scattered if for every nonempty closed subset A of X, there is a point $x \in A$ which has a Čech-complete neighborhood in A. They proved

(E) (Hohti and Ziqui [9]) If $\{X_n: n \in \omega\}$ is a countable collection of Č-scattered paracompact spaces, then the product $\prod_{n\in\omega} X_n$ is paracompact.

It seems to be natural to consider the paracompactness (Lindelöf property) of product of a perfect paracompact (hereditarily Lindelöf) space and a product of countably many Čech-scattered paracompact (Lindelöf) spaces. So, we prove analogous results of (B) and (C).

All spaces are assumed to be Tychonoff spaces. Let ω denote the set of natural numbers. Undefined terminology can be found in Engelking [5].

2. Preliminaries

Let X be a space. For a closed subset A of X, let

$$A^* = \{x \in A: x \text{ has no Čech complete neighborhood in } A\}.$$

Let $A^{(0)} = A$, $A^{(\alpha+1)} = (A^{(\alpha)})^*$ and $A^{(\alpha)} = \bigcap_{\beta<\alpha} A^{(\beta)}$ for a limit ordinal α. Note that every $A^{(\alpha)}$ is a closed subset of X and if A and B are closed subsets of X such that $A \subset B$, then $A^{(\alpha)} \subset B^{(\alpha)}$ for each ordinal α. Furthermore, X is Čech-scattered if and only if $X^{(\alpha)} = \emptyset$ for some ordinal α. It is clear that if X is a Čech-scattered space and A is an open (closed) subset of X, then A is also Čech-scattered. A subset A of X is said to be *topped* if there is an ordinal $\alpha(A)$ such that $A^{(\alpha(A))}$ is nonempty and Čech-complete. For each $x \in X$, there is a unique ordinal α such that $x \in X^{(\alpha)} - X^{(\alpha+1)}$, which is denoted by $\mathrm{rank}(x) = \alpha$. Then there is a neighborhood base $\mathcal{B}$ of x in X, consisting of open subsets of X, such that for each $B \in \mathcal{B}$, $cl B$ is topped in X and $\alpha(\mathrm{cl}\, B) = \mathrm{rank}(x)$.

The proofs of following lemmas are routine. So we omit them.

Lemma 2.1. (1) *If X and Y are Čech-scattered spaces, then the product $X \times Y$ is Čech-scattered.*

(2) *Let X and Y be spaces and $f: X \to Y$ be a perfect mapping from X onto Y. Then X is Čech-scattered if and only if Y is Čech-scattered.*