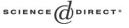


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# The natural mappings $i_n$ and k-subspaces of free topological groups on metrizable spaces

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#### Abstract

Let F(X) be the free topological group on a Tychonoff space X. For all natural number n we denote by  $F_n(X)$  the subset of F(X) consisting of all words of reduced length  $\leq n$ , and by  $i_n$  the natural mapping from  $(X \oplus X^{-1} \oplus \{e\})^n$  to  $F_n(X)$ . We prove that for a metrizable space X if  $F_n(X)$ is a k-space for each n, then X is locally compact and either separable or discrete. Therefore, as a corollary, we obtain that for a metrizable space X if  $F_n(X)$  is a k-space for all  $n \in \mathbb{N}$ , then so is F(X). Furthermore, it is proved that for a metrizable space X the following are equivalent: (i) the mapping  $i_n$  is a quotient mapping for each n; (ii) a subset U of F(X) is open if  $i_n^{-1}(U \cap F_n(X))$  is open in  $(X \oplus X^{-1} \oplus \{e\})^n$  for each n; (iii) X is locally compact separable or discrete. © 2004 Elsevier B.V. All rights reserved.

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#### 1. Introduction

Let F(X) and A(X) be respectively the *free topological group* and the *free Abelian* topological group on a Tychonoff space X in the sense of Markov [7]. As an abstract group, F(X) is free on X and it carries the finest group topology that induces the original topology of X, in other words, every continuous map from X to an arbitrary topological group lifts in a unique fashion to a continuous homomorphism from F(X). Similarly, as an abstract group, A(X) is the free Abelian group on X, having the finest group topology that

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induces the original topology of X, so that every continuous map from X to an arbitrary Abelian topological group extends to a unique continuous homomorphism from A(X).

For each  $n \in \mathbb{N}$ ,  $F_n(X)$  stands for a subset of F(X) formed by all words whose length is less than or equal to n. It is known that X itself and each  $F_n(X)$  are closed in F(X). The subspace  $A_n(X)$  is defined similarly and each  $A_n(X)$  is closed in A(X). Let e be the identity of F(X) and 0 be that of A(X). For each  $n \in \mathbb{N}$  and an element  $(x_1, x_2, \ldots, x_n)$  of  $(X \oplus X^{-1} \oplus \{e\})^n$  we call  $x_1x_2 \cdots x_n$  a form. In the Abelian case,  $x_1 + x_2 + \cdots + x_n$  is also called a form for  $(x_1, x_2, \ldots, x_n) \in (X \oplus -X \oplus \{0\})^n$ . We remark that a form may contain some reduced letter. Then the reduced form of  $x_1x_2 \cdots x_n$  is a word of F(X) and that of  $x_1 + x_2 + \cdots + x_n$  is a word of A(X). For each  $n \in \mathbb{N}$  we denote the natural mapping from  $(X \oplus X^{-1} \oplus \{e\})^n$  onto  $F_n(X)$  by  $i_n$  and we also use the same symbol  $i_n$  in the Abelian case, that is,  $i_n$  means the natural mapping from  $(X \oplus -X \oplus \{0\})^n$  onto  $A_n(X)$ . Clearly the mapping  $i_n$  is continuous for each  $n \in \mathbb{N}$ .

The following problems have been studied by several mathematicians and described in [10].

**Problem 1.** Characterize spaces *X* for which the mapping  $i_n$  is a quotient (closed, *z*-closed, *R*-quotient, etc.) mapping for all  $n \in \mathbb{N}$ .

**Problem 2.** Find general conditions on *X* implying that F(X) (or  $F_n(X)$  for each  $n \in \mathbb{N}$ ) is a *k*-space.

Problem 1 was completely solved for n = 2 by Pestov [8]. He proved that  $i_2$  is a quotient mapping iff X is strongly collectionwise normal, i.e., if every neighborhood of the diagonal in  $X^2$  contains a uniform neighborhood of the diagonal. Furthermore, the author [13] proved that  $i_2$  is a quotient mapping iff  $i_2$  is closed. The author also proved in the same paper that for a metrizable space X the mapping  $i_n$  is closed for each  $n \in \mathbb{N}$  iff X is compact or discrete. They are true for Abelian case.

The author [12] obtained a characterization of a metrizable space such that every  $i_n$  is a quotient mapping for Abelian case. He proved that for a metrizable space X,  $i_n$  for Abelian case is a quotient mapping for each  $n \in \mathbb{N}$  if and only if either X is locally compact and the set dX of all nonisolated points in X is separable, or dX is compact. As the author mentioned in [12, Proposition 4.1], for a Dieudonné complete (and hence, metrizable) space X,  $i_n$  is a quotient mapping iff  $A_n(X)$  ( $F_n(X)$ ) is a k-space for each  $n \in \mathbb{N}$ . So, the above result is also an answer to Problem 2 for the free Abelian topological group on a metrizable space.

The aim of this paper is to solve the above problems for the *non-Abelian* free topological group on a metrizable space. As a consequence, we can know whether each  $i_n : (X \oplus X^{-1} \oplus \{e\})^n \to F_n(X)$  is a quotient mapping or not, and hence whether each  $F_n(X)$  is a *k*-space or not for such familiar metric spaces *X* as the real line  $\mathbb{R}$ , the space  $\mathbb{Q}$  of rational numbers,  $\mathbb{R} \setminus \mathbb{Q}$ ,  $J(\kappa)$  ( $\kappa \ge \omega$ ) be the hedgehog space of spine  $\kappa$  such that each spine is a sequence which converges to the center point or the topological sum  $C_{\kappa}$  of  $\kappa (\ge \omega)$  many convergent sequences with their limits.

We first show that for a metrizable space X if  $i_n$  for non-Abelian case is a quotient mapping for each  $n \in \mathbb{N}$ , then X is locally compact separable or discrete. Then we shall

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