Available online at www.sciencedirect.com
SCIENCE DIRECT
Topology and its Applications 146–147 (2005) 239–251
Topology and its Applications
www.elsevier.com/locate/topol

The natural mappings i_n and k-subspaces of free topological groups on metrizable spaces

Kohzo Yamada

Department of Mathematics, Faculty of Education, Shizuoka University, Shizuoka 422-8529, Japan

Received 12 September 2002; received in revised form 27 February 2003

Dedicated to Professor Takao Hoshina on his sixtieth birthday

Abstract

Let $F(X)$ be the free topological group on a Tychonoff space X. For all natural number n we denote by $F_n(X)$ the subset of $F(X)$ consisting of all words of reduced length $\leqslant n$, and by i_n the natural mapping from $(X \oplus X^{-1} \oplus \{e\})^n$ to $F_n(X)$. We prove that for a metrizable space X if $F_n(X)$ is a k-space for each n, then X is locally compact and either separable or discrete. Therefore, as a corollary, we obtain that for a metrizable space X if $F_n(X)$ is a k-space for all $n \in \mathbb{N}$, then so is $F(X)$. Furthermore, it is proved that for a metrizable space X the following are equivalent: (i) the mapping i_n is a quotient mapping for each n; (ii) a subset U of $F(X)$ is open if $i_n^{-1}(U \cap F_n(X))$ is open in $(X \oplus X^{-1} \oplus \{e\})^n$ for each n; (iii) X is locally compact separable or discrete.

MSC: primary 54H11, 22A05, 54C10, 54D50; secondary 54E35, 54D45, 54D65

Keywords: Free topological group; Free Abelian topological group; Quotient mapping; Metrizable space; Locally compact; Separable

1. Introduction

Let $F(X)$ and $A(X)$ be respectively the *free topological group* and the *free Abelian topological group* on a Tychonoff space X in the sense of Markov [7]. As an abstract group, $F(X)$ is free on X and it carries the finest group topology that induces the original topology of X, in other words, every continuous map from X to an arbitrary topological group lifts in a unique fashion to a continuous homomorphism from $F(X)$. Similarly, as an abstract group, $A(X)$ is the free Abelian group on X, having the finest group topology that

E-mail address: eckyama@ipc.shizuoka.ac.jp (K. Yamada).

0166-8641/$ – see front matter
doi:10.1016/j.topol.2003.02.011

induces the original topology of X, so that every continuous map from X to an arbitrary Abelian topological group extends to a unique continuous homomorphism from $A(X)$.

For each $n \in \mathbb{N}$, $F_n(X)$ stands for a subset of $F(X)$ formed by all words whose length is less than or equal to n. It is known that X itself and each $F_n(X)$ are closed in $F(X)$. The subspace $A_n(X)$ is defined similarly and each $A_n(X)$ is closed in $A(X)$. Let e be the identity of $F(X)$ and 0 be that of $A(X)$. For each $n \in \mathbb{N}$ and an element $(x_1, x_2, \ldots, x_n)$ of $(X \oplus X^{-1} \oplus \{e\})^n$ we call $x_1 x_2 \cdots x_n$ a *form*. In the Abelian case, $x_1 + x_2 + \cdots + x_n$ is also called a *form* for $(x_1, x_2, \ldots, x_n) \in (X \oplus -X \oplus \{0\})^n$. We remark that a form may contain some reduced letter. Then the reduced form of $x_1 x_2 \cdots x_n$ is a word of $F(X)$ and that of $x_1 + x_2 + \cdots + x_n$ is a word of $A(X)$. For each $n \in \mathbb{N}$ we denote the natural mapping from $(X \oplus X^{-1} \oplus \{e\})^n$ onto $F_n(X)$ by i_n and we also use the same symbol i_n in the Abelian case, that is, i_n means the natural mapping from $(X \oplus -X \oplus \{0\})^n$ onto $A_n(X)$. Clearly the mapping i_n is continuous for each $n \in \mathbb{N}$.

The following problems have been studied by several mathematicians and described in [10].

Problem 1. Characterize spaces X for which the mapping i_n is a quotient (closed, z-closed, R-quotient, etc.) mapping for all $n \in \mathbb{N}$.

Problem 2. Find general conditions on X implying that $F(X)$ (or $F_n(X)$ for each $n \in \mathbb{N}$) is a k-space.

Problem 1 was completely solved for $n = 2$ by Pestov [8]. He proved that i_2 is a quotient mapping iff X is strongly collectionwise normal, i.e., if every neighborhood of the diagonal in X^2 contains a uniform neighborhood of the diagonal. Furthermore, the author [13] proved that i_2 is a quotient mapping iff i_2 is closed. The author also proved in the same paper that for a metrizable space X the mapping i_n is closed for each $n \in \mathbb{N}$ iff X is compact or discrete. They are true for Abelian case.

The author [12] obtained a characterization of a metrizable space such that every i_n is a quotient mapping for Abelian case. He proved that for a metrizable space X, i_n for Abelian case is a quotient mapping for each $n \in \mathbb{N}$ if and only if either X is locally compact and the set dX of all nonisolated points in X is separable, or dX is compact. As the author mentioned in [12, Proposition 4.1], for a Dieudonné complete (and hence, metrizable) space X, i_n is a quotient mapping iff $A_n(X)$ ($F_n(X)$) is a k-space for each $n \in \mathbb{N}$. So, the above result is also an answer to Problem 2 for the free Abelian topological group on a metrizable space.

The aim of this paper is to solve the above problems for the *non-Abelian* free topological group on a metrizable space. As a consequence, we can know whether each $i_n : (X \oplus X^{-1} \oplus \{e\})^n \to F_n(X)$ is a quotient mapping or not, and hence whether each $F_n(X)$ is a k-space or not for such familiar metric spaces X as the real line $\mathbb{R}$, the space $\mathbb{Q}$ of rational numbers, $\mathbb{R} \setminus \mathbb{Q}$, $J(\kappa)$ ($\kappa \geqslant \omega$) be the hedgehog space of spine κ such that each spine is a sequence which converges to the center point or the topological sum C_κ of $\kappa\,(\geqslant \omega)$ many convergent sequences with their limits.

We first show that for a metrizable space X if i_n for non-Abelian case is a quotient mapping for each $n \in \mathbb{N}$, then X is locally compact separable or discrete. Then we shall