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$C_p(X)$ in coreflective classes of locally convex spaces

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Abstract

Situations analogous to some classical characterization are investigated, of topological spaces X for which $C_p(X)$ belongs to a given coreflective class $\mathcal{C}$ of locally convex spaces. For instance, if $\mathcal{C}$ contains all strong Mazur spaces and is contained in the class of weak Mazur spaces, then $C_p(X)$ belongs to $\mathcal{C}$ iff X is realcompact. If $\mathcal{C}$ is the coreflective hull of $\mathbb{R}^\omega$ and X is a P-space, then $C_p(X)$ belongs to $\mathcal{C}$ iff X is realcompact.

MSC: 54C35; 46E10

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In the present paper, $C_p(X)$ denotes the locally convex space of real-valued continuous functions on a Tychonoff space X, where both the linear and topological structures are inherited from the canonical embedding of $C_p(X)$ into $\mathbb{R}^X$.

Recall the following results for locally convex spaces (LCS):

Theorem 1 (Mrówka [16], V. Pták (unpublished)). *$C_p(X)$ is a Mazur space* (*i.e., linearly sequential*) *iff X is realcompact.*

Theorem 2 (Buchwalter and Schmets [4]). *$C_p(X)$ is barrelled iff every relatively pseudocompact subset of X is finite.*

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Theorem 3 (Schmets [18]). *$C_p(X)$ is a bornological space iff X is realcompact.*

Similar results are known for some other classes of LCS like quasi-barrelled or ultrabornological spaces (see, e.g., [18,6]). All the mentioned classes of locally convex spaces form coreflective classes in LCS and so, a general question may be

Question 1. For a given coreflective class $\mathcal{C}$ in LCS characterize those topological spaces X with $C_p(X) \in \mathcal{C}$.

In two of the above theorems, realcompactness of X characterizes the required properties of $C_p(X)$. So, we may ask a "subquestion":

Question 2. For what coreflective classes $\mathcal{C}$ in LCS, $C_p(X) \in \mathcal{C}$ iff X is realcompact?

We shall give partial answers to both questions. In some situations, this investigation brings new looks or unified proofs. Complete answers for discrete spaces X to both previous questions are given in [10] (see the next part of this introduction).

We shall now recall some concepts and terminology. Most of the concepts and terms used in this paper come from the books [12,20] (topological linear spaces), [7] (general topology) and [1] (category theory). We want to specify at this place that all the topological spaces considered are Tychonoff.

A nonzero cardinal κ is called *measurable* if there exists a nontrivial κ-additive two-valued measure on κ vanishing on singletons (κ-additivity of μ means that $\mu(\bigcup_I A_i) = \sum_I \mu(A_i)$ for every disjoint family $\{A_i\}_I$ in κ with $|I| < \kappa$). Realize that ω is measurable by our definition (it seems to be convenient for formulations of our results to include ω among measurable cardinals).

We shall index measurable cardinals by ordinals: $\mathfrak{m}_\alpha$ is the αth measurable cardinal. Thus $\mathfrak{m}_0 = \omega_0$ and $\mathfrak{m}_1$ is the usual Ulam measurable cardinal (all cardinals less than $\mathfrak{m}_1$ are called Ulam nonmeasurable). The cardinal $\mathfrak{m}_{\alpha+1}$ is the first cardinal admitting a nontrivial $\mathfrak{m}_\alpha^+$-additive two-valued measure being zero on points. It is known that $\kappa < \mathfrak{m}_{\alpha+1}$ iff every ultrafilter on κ that is closed under $\mathfrak{m}_\alpha$ intersections, has a nonempty intersection; in other words, iff a discrete space of cardinality κ is $\mathfrak{m}_\alpha$-compact, i.e., can be embedded as a closed subspace into a product of subspaces of Tychonoff cubes of weight at most $\mathfrak{m}_\alpha$. The analogous characterization of $\kappa < \mathfrak{m}_\alpha$ for a limit α uses $\sup\{\mathfrak{m}_\beta\colon \beta < \alpha\}$ instead of $\mathfrak{m}_{\alpha-1}$.

It is convenient to use the language of category theory for our investigation. Every subcategory will be full and so it suffices to speak about subclasses of objects instead of subcategories. We shall work in the category LCS of locally convex topological linear spaces over $\mathbb{R}$ and continuous linear maps.

Since $\mathbb{R}$ is a retract of $C_p(X)$ whenever X is nonempty and we are interested in those coreflective classes containing $C_p(X)$, we shall always assume that our coreflective classes $\mathcal{C}$ contain $\mathbb{R}$ or, equivalently, that $\mathcal{C}$ are bicoreflective. We are thus avoiding classes composed of spaces having zero dual. Bicoreflectivity means that the coreflective maps are linear isomorphisms, i.e., that for every space $X \in$ LCS there exists a finer space cX belonging to $\mathcal{C}$ such that every continuous linear mapping from a space in $\mathcal{C}$ to X is continuous already into the finer space cX. Equivalently, $\mathcal{C}$ are closed under inductive