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Codimension one embeddings of product of three spheres

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Abstract

Let $f:S^p \times S^q \times S^r \to S^{p+q+r+1}$ be a smooth embedding with $1 \leqslant p \leqslant q \leqslant r$. For $p \geqslant 2$, the authors have shown that if $p+q \neq r$, or $p+q=r$ and r is even, then the closure of one of the two components of $S^{p+q+r+1} - f(S^p \times S^q \times S^r)$ is diffeomorphic to the product of two spheres and a disk, and that otherwise, there are infinitely many embeddings, called exotic embeddings, which do not satisfy such a property. In this paper, we study the case $p=1$ and construct infinitely many exotic embeddings. We also give a positive result under certain (co)homological hypotheses on the complement. Furthermore, we study the case $(p,q,r)=(1,1,1)$ more in detail and show that the closures of the two components of $S^4 - f(S^1 \times S^1 \times S^1)$ are homeomorphic to the exterior of an embedded solid torus or Montesinos' twin in S^4.

MSC: primary 57R40; secondary 57Q45

Keywords: Embedding; Product of spheres; Torus theorem; Codimension one; Twin

1. Introduction

Let $f:S^p \times S^q \times S^r \to S^{p+q+r+1}$ be a smooth embedding with $1 \leqslant p \leqslant q \leqslant r$. For $p \geqslant 2$, the authors [10] have shown that if $p+q \neq r$, or $p+q=r$ and r is even, then the closure of one of the two components of $S^{p+q+r+1} - f(S^p \times S^q \times S^r)$ is

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diffeomorphic to $S^p \times S^q \times D^{r+1}$, $S^p \times D^{q+1} \times S^r$ or $D^{p+1} \times S^q \times S^r$. Furthermore, the condition on p, q and r is essential; i.e., if it is not satisfied, then there exist infinitely many embeddings without such a property. This last result is very surprising, since codimension one embeddings of a sphere or a product of two spheres always have the expected property as mentioned above with possible exceptions in certain dimensions involving the (generalized) Poincaré conjecture where "diffeomorphic" should be replaced by "homeomorphic" (see [1,8,16,6,14,9]). For this reason, we say that an embedding f is *exotic* if the closure of neither of the two components of $S^{p+q+r+1} - f(S^p \times S^q \times S^r)$ is diffeomorphic to a product of two spheres and a disk.

The purpose of this paper is to study the case $p = 1$. First we prove the following.

Theorem 1.1. *If $1 \leqslant q \leqslant r$, then there exist infinitely many mutually distinct embeddings $f_n : S^1 \times S^q \times S^r \to S^{q+r+2}$, $n = 1, 2, 3, \ldots$, such that the closure of neither of the two components of $S^{q+r+2} - f_n(S^1 \times S^q \times S^r)$ is homotopy equivalent to a product of spheres.*

Combining the above theorem with the above mentioned result for the case $p \geqslant 2$, we obtain a complete characterization of those triples (p, q, r) for which exotic embeddings $f : S^p \times S^q \times S^r \to S^{p+q+r+1}$ exist.

When $p = 1$, under certain conditions, we have the following affirmative result as well. Note that for a smooth embedding $f : S^p \times S^q \times S^r \to S^{p+q+r+1}$ with p, q and r arbitrary, it is not difficult to show that one of the two components of $S^{p+q+r+1} - f(S^p \times S^q \times S^r)$ always has the homology of $S^p \times S^q$ or $S^p \times S^r$ or $S^q \times S^r$ (for details see [10, Lemma 2.1]).

Proposition 1.2. *Let $f : S^1 \times S^q \times S^r \to S^{q+r+2}$, $2 \leqslant q \leqslant r$, be a smooth embedding and C_1 the closure of one of the two components of $S^{q+r+2} - f(S^1 \times S^q \times S^r)$.*

- (i) *Suppose $r \neq q + 1$. If $H_*(C_1; \mathbb{Z}) \cong H_*(S^q \times S^r; \mathbb{Z})$, then C_1 is diffeomorphic to $D^2 \times S^q \times S^r$.*
- (ii) *Suppose $r = q + 1$. If C_1 has the same cohomology ring as $S^q \times S^r$, then C_1 is diffeomorphic to $D^2 \times S^q \times S^r$.*

We will show that the conditions $q \geqslant 2$ and $r \neq q + 1$ are essential in Proposition 1.2(i). We will also study the case where C_1 has the homology of $S^1 \times S^q$ or $S^1 \times S^r$.

The second purpose of this paper is to study embeddings of $T^3 = S^1 \times S^1 \times S^1$ in S^4 more in detail. The most standard way to embed T^3 into S^4 is to first embed the 2-dimensional torus T^2 standardly into S^4 and take the boundary of its tubular neighborhood. For such an embedding, the closure of one of the two components of the complement is of course diffeomorphic to $T^2 \times D^2$, while the closure of the other one is known as Montesinos' twin [12], denoted by *Tw*. Recall that *Tw* is a regular neighborhood of the union of two embedded 2-spheres in S^4 intersecting each other transversely at two points with opposite signs.

In [7, Example 4.6], it has been shown that there are plenty of smooth embeddings of T^3 in S^4. In fact, there are embeddings such that the closure of neither of the two components of the complement is homeomorphic to $T^2 \times D^2$ or *Tw*.