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Analytic k-spaces

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Abstract

We study sequential convergence in spaces with analytic topologies avoiding thus a number of standard pathologies. For example, we identify bisequentiality of an analytic space as the Fréchet property of its square. We show that a countable Fréchet group is metrizable if and only if its topology is analytic. We also investigate the diagonal sequence properties and show their productiveness in the class of analytic spaces.

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1. Introduction

This is a continuation of our paper [23] where we study effective versions of some standard topological problems and results. Thus we restrict ourselves to (regular) countable topological spaces X with the property that the family τ_X of all open subsets of X is in some sense effective. For example, if τ_X as a subset of the Cantor cube 2^X is a continuous image of the irrationals then we call any such X an *analytic space*. Many of the standard examples of countable spaces are analytic. For example, the *Arens space* [1], the *Arhangel'ski–Franklin space* [3], and the countable *sequential fan* [14] are all analytic spaces. On the other hand, many topological applications to the study of, say, weak topologies of Banach spaces require results about countable analytic spaces (see, e.g., [2]).

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Recall, that X is said to be a *k-space* if and only if an arbitrary subset of X is closed just in case its intersection with an arbitrary compact subset of X is closed (see, e.g., [17]). In our context this reduces to the more familiar class of *sequential spaces*. Recall that X is said to be sequential if for every nonclosed $A \subseteq X$ there is a sequence of elements of A converging to a point outside of A. If we require a sequence of elements of A to converge to an arbitrary point of the closure of A we get the considerably more restrictive class of *Fréchet spaces*. It is usually in relation to these classes of spaces that one considers various ways to obtain a converging sequence out of a sequence of converging sequences. Recall that the *diagonal-sequence property* states that if $\{x_{nk}\}$ is a double-indexed sequence of members of X such that for some $x \in X$ and all n, $x_{nk} \to_k x$ then for each n we can choose $k(n)$ such that $x_{nk(n)} \to_n x$. If we require that some infinite subsequence of $\{x_{nk(n)}\}$ converges to x rather than the sequence itself, we get the *weak diagonal sequence property*. Note that the diagonal sequence property and the weak diagonal sequence property are formally incomparable with the Fréchet property. Consider, for example, Arens space and the Sequential fan. The former has the diagonal sequence property but it is not Fréchet while the later is Fréchet but it fails even the weak diagonal sequence property. It turns out that in the context of analytic spaces the diagonal sequence property is as restrictive as first countability (metrizability) (see [20,23]). We give here a variation of this result by showing that an analytic sequential space with the diagonal sequence property is weakly first countable. Recall that we say that X is *weakly first countable* if for every $x \in X$ we can find a decreasing sequence of sets $B(x,n) \ni x$ such that a set V is open iff for all $x \in V$, there is m with $B(x,m) \subseteq V$. Note that the topology of an arbitrary countable weakly first countable space is analytic (in fact, $F_{\sigma\delta}$). For example, the Arens space is a typical example of a weakly first countable space. Note that every weakly first countable space has the diagonal sequence property and that every Fréchet weakly first countable space is in fact first countable.

Recall now that X is said to be *bisequential* if for every ultrafilter $\mathcal{U}$ over X converging to some point x there is a sequence $A_n \in \mathcal{U}$ converging to x. Clearly every bisequential space is Fréchet but not vice versa. Consider, for example, the Sequential fan. Note also that every bisequential space has the weak diagonal sequence property but not vice versa. Consider, for example, the Arens space. We show however that the two properties jointly characterize bisequentiality in the class of analytic space. Thus we show that every analytic Fréchet space with the weak diagonal sequence property is bisequential. We give some application of this result to the study of products as well as to the study of countable topological groups. For example, we show that the square of an analytic Fréchet space X is Fréchet if and only if X contains no copy of the sequential fan $S(\omega)$. As another application we show that analytic Fréchet groups are metrizable solving thus the effective version of the well-known problem of Malyhin (see, e.g., [14]). The preservation of the weak diagonal sequence property in products of analytic spaces seems curiously related to the problem whether the Sequential fan is a test space for the failure of this property in the class of analytic spaces. This can be seen from the fact which we show here which says that the sequential fan does not embed into the product of two analytic spaces with the weak diagonal sequence property. This can be regarded as a proof of the effective version of a conjecture of Nogura [11]. The proof of the unrestricted version of Nogura's conjecture is given in [22] and our proof here can be regarded as its effective version.