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Countable compactness and finite powers of topological groups without convergent sequences ☆

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Abstract

We show under $\mathrm{MA}_{\mathrm{countable}}$ that for every positive integer n there exists a topological group G without non-trivial convergent sequences such that G^n is countably compact but G^{n+1} is not. This answers the finite case of Comfort's Question 477 in the Open Problems in Topology. We also show under $\mathrm{MA}_{\mathrm{countable}} + 2^{<\mathfrak{c}} = \mathfrak{c}$ that there are $2^{\mathfrak{c}}$ non-homeomorphic group topologies as above if $n \geqslant 2$. We apply the construction on suitable sets, answering the finite case of a question of D. Dikranjan on the productivity of suitability and in a topological game defined by Bouziad.
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1. Introduction

One of the best well-known results in topology is Tychonoff's theorem on the productivity of compactness. A natural question is whether other compact-like properties are productive as well. There are countably compact spaces whose square are not even pseudocompact (due independently to Novák [21] and Terasaka [25]). Frolík [11] extended this study to finite products, showing that for every n there exists a space X such that

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X^n is countably compact but X^{n+1} is not. On the other hand, Scarborough and Stone [24] showed that if the $2^{2^{\mathfrak{c}}}$th power of a space is countably compact then every power is countably compact. This last result was improved by Ginsburg and Saks [14] who showed that a space is countably compact for every power if and only if its $2^{\mathfrak{c}}$th power is countably compact. The natural question whether $2^{\mathfrak{c}}$ was the best possible was answered under MA by Saks [23]. Recently, using an equivalence of Yang [30], M. Hrusak noted that in Shelah's model [2], the best possible cardinal in Ginsburg and Saks theorem is not larger than $\mathfrak{c}$.

Topological groups is an important class of spaces in which productivity has been studied, resulting in interesting theorems and examples. A quite unexpected result was obtained by Comfort and Ross [6], who showed that pseudocompactness becomes productive for the class of topological groups. A natural question, asked by Comfort, was whether the same would be true for countably compact topological groups. A consistent negative answer was obtained by van Douwen [10] who showed under MA that there exist two countably compact topological groups whose product is not countably compact. A decade later, Hart and van Mill [17] showed that there exists under $\mathrm{MA}_{\mathrm{countable}}$ a countably compact topological group whose square is not countably compact.

Because of these results, Comfort asked the following question in the Open Problems in Topology:

Question 1.1 [4]. *Is there, for every* (*not necessarily infinite*) *cardinal number* $\alpha \leqslant 2^{\mathfrak{c}}$, *a topological group G such that G^γ is countably compact for all cardinals $\gamma < \alpha$, but G^α is not countably compact*?

Under $\mathrm{MA}_{\mathrm{countable}}$, it was shown that two [17] and three [28] are such cardinals. Furthermore, there are infinitely many such natural numbers [27]. All the examples above contain convergent sequences.

We answer Comfort's question for the finite case, under $\mathrm{MA}_{\mathrm{countable}}$, providing witnesses without non-trivial convergent sequences. In addition, if $2^{<\mathfrak{c}} = \mathfrak{c}$ is also assumed, then $2^{\mathfrak{c}}$ non-homeomorphic examples can be obtained for $n \geqslant 3$. Some kind of 'basis property' for sequences of length at most n are used to obtain the construction. The motivation for this approach came from Steve Watson's lecture on Kunen's solution to a question of van Douwen on Bohr topologies [20], given at University of São Paulo.

In [7], Dikranjan studied cardinal numbers related to the productivity of a topological property, restating Comfort's question in this setting. He also asked about the productivity of suitability, which was introduced by Hofmann and Morris. This concept, introduced in [18], is a natural generalization of monotheticity.

Definition 1.2. A subset S of a topological group H is a (closed) suitable set for H if (a) S is discrete in H and (b) S is closed in $H \setminus \{e\}$ (respectively H) and (c) the group generated by S is dense in H.

A topological group is monothetic if it has a suitable set of size 1. Hofmann and Morris showed that every locally compact group has a suitable set; in [5] the study of the non-locally compact case was started and it was followed in [8,9,22,26,29]. It is easy to see that if a power of H has a suitable set then every larger power also does. Dikranjan [7] asked

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