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Polyhedra dominating finitely many different homotopy types

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Abstract

In 1968 K. Borsuk asked: Is it true that every finite polyhedron dominates only finitely many different shapes? In this question the notions of shape and shape domination can be replaced by the notions of homotopy type and homotopy domination.

We obtained earlier a negative answer to the Borsuk question and next results that the examples of such polyhedra are not rare. In particular, there exist polyhedra with nilpotent fundamental groups dominating infinitely many different homotopy types. On the other hand, we proved that every polyhedron with finite fundamental group dominates only finitely many different homotopy types. Here we obtain next positive results that the same is true for some classes of polyhedra with Abelian fundamental groups and for nilpotent polyhedra. Therefore we also get that every finitely generated, nilpotent torsion-free group has only finitely many r-images up to isomorphism.

MSC: 55P55; 55P15

Keywords: Polyhedron; CW-complex; Homotopy domination; Homotopy type; Compactum; Shape; Shape domination

An introduction

By a polyhedron we mean, as usual, a finite one. Each polyhedron and CW-complex is assumed to be connected (for convenience).

In 1968 at the Topological Conference in Herceg-Novi K. Borsuk stated the following:

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Problem. Does every polyhedron dominate only finitely many different shapes?

See [2]; see also [1] for an equivalent version of the question, for *FANR*'s.
(The basic notions and results of shape theory the reader can find in [3,18,7].)

In the above problem the notions of shape and shape domination can be replaced by the notions of homotopy type and homotopy domination (respectively). Indeed, by the known results in shape theory (from [8,10], [7, Theorems 2.2.6, and 2.1.6]) we get that, for each polyhedron P, there is a 1–1 functorial correspondence between the shapes of compacta shape dominated by P and the homotopy types of CW-complexes (not necessarily finite) homotopy dominated by P (in both pointed and unpointed cases).

In the sequel we will concentrate on the pointed version of the question. By the results of [10] and [6, Theorem 5.1] we obtain that in pointed and unpointed cases the answer is the same.

Recall that each space homotopy dominated by a polyhedron has the homotopy type of a CW-complex, not necessarily finite (1949 J.H.C. Whitehead; see also [24]). Thus the Borsuk problem is equivalent to:

Problem. Does every polyhedron homotopy dominate only finitely many different homotopy types?

In this paper we consider dominations of a polyhedron in the category of CW-complexes and homotopy classes of cellular maps between them.

When the question was stated, it was known that every 1-dimensional polyhedron dominates only finitely many different shapes. It is a consequence of the result of S. Trybulec [23] that every movable curve has a plane shape and the theorem of K. Borsuk stating that two plane continua have the same shape if and only if their Betti numbers coincide [3, Theorem 7.1, p. 221].

By the result of S. Mather [17] (see also a simple paper of Holsztyński [12]), every polyhedron dominates only a countable number of different homotopy types (hence shapes). Directly in shape category the same was proven by Moron and Ruiz del Portal in [19].

In [16] we showed that generally an answer to the Borsuk question is negative: there exists a polyhedron (even of dimension 2), which homotopy dominates infinitely many polyhedra of different homotopy types.

Moreover, we proved that such examples are not rare: for every non-Abelian poly-$\mathbb{Z}$-group G and an integer $n \geqslant 3$ there exists a polyhedron P with $\pi_1(P) \cong G$ and $\dim P = n$ dominating infinitely many polyhedra of different homotopy types (see [13]). Thus, there exist polyhedra with nilpotent fundamental groups with this property.

On the other hand, in [15] we obtained, using the results of localization theory in the homotopy category of CW-complexes, that every simply-connected polyhedron dominates only finitely many different homotopy types. In [14] we proved, in an other way, that polyhedra with finite fundamental groups dominate only finitely many different homotopy types.