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Compositions of theta correspondences

Hongyu He

Department of Mathematics & Statistics, Georgia State University, Atlanta, GA 30303, USA

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Dedicated to the memory of my father, Decai He

Abstract

Theta correspondence θ over $\mathbb{R}$ is established by Howe (J. Amer. Math. Soc. 2 (1989) 535). In He (J. Funct. Anal. 199 (2003) 92), we prove that θ preserves unitarity under certain restrictions, generalizing the result of Li (Invent. Math. 97 (1989) 237). The goal of this paper is to elucidate the idea of constructing unitary representation through the propagation of theta correspondences. We show that under a natural condition on the sizes of the related dual pairs which can be predicted by the orbit method (J. Algebra 190 (1997) 518; Representation Theory of Lie Groups, Park City, 1998, pp. 179–238; The Orbit Correspondence for real and complex reductive dual pairs, preprint, 2001), one can compose theta correspondences to obtain unitary representations. We call this process quantum induction.

MSC: primary 22E45; 22E46

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1. Introduction

An important problem in representation theory is the classification and construction of irreducible unitary representations. Let G be a reductive group and $\Pi(G)$ be its admissible dual. For an algebraic semisimple group G, the admissible dual $\Pi(G)$ is known, mostly due to the works of Harish-Chandra, R. Langlands, and Knapp–Zuckerman (see [17,18]). Let $\Pi_{\mathrm{u}}(G)$ be the set of

E-mail address: hhe@gsu.edu.

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equivalence classes of irreducible unitary representations of G, often called the unitary dual of G. The unitary dual of general linear groups is classified by Vogan [29]. The unitary dual of complex classical groups is classified by Barbasch [2]. Recently, Barbasch has classified all the spherical duals for split classical groups (see [3]). The unitary duals $\Pi_{\mathrm{u}}(O(p,q))$ and $\Pi_{\mathrm{u}}(Sp_{2n}(\mathbb{R}))$ are not known in general.

In [14], Howe constructs certain small unitary representations of the symplectic group using Mackey machine. Later, Jian-Shu Li generalizes Howe's construction of small unitary representations to all classical groups. In particular, Li defines a sesquilinear form $(,)_\pi$ that relates these constructions to the theta correspondence (see [11,20]). It then becomes clear to many people that some irreducible unitary representations can be constructed through the propagation of theta correspondences (see [15,21,28] and the references within them). So far, constructions can only be carried out for "complete small orbits" (see [21]). The purpose of this paper is to make it work for nilpotent orbits in general, for real orthogonal groups and symplectic groups.

Consider the group $O(p,q)$ and $Sp_{2n}(\mathbb{R})$. The theta correspondence with respect to

$$O(p,q) \to Sp_{2n}(\mathbb{R})$$

is formulated by Howe as a one-to-one correspondence

$$\theta(p,q;2n) : \mathscr{R}(MO(p,q), \omega(p,q;2n)) \to \mathscr{R}(MSp_{2n}(\mathbb{R}), \omega(p,q;2n)),$$

where $MO(p,q)$ and $MSp_{2n}(\mathbb{R})$ are some double coverings of $O(p,q)$ and $Sp_{2n}(\mathbb{R})$, respectively, and

$$\mathscr{R}(MO(p,q), \omega(p,q;2n)) \subseteq \Pi(MO(p,q)),$$

$$\mathscr{R}(MSp_{2n}(\mathbb{R}), \omega(p,q;2n)) \subseteq \Pi(MSp_{2n}(\mathbb{R}))$$

(see [13]). We denote the inverse of $\theta(p,q;2n)$ by $\theta(2n;p,q)$. For the sake of simplicity, we define

$$\theta(p,q;2n)(\pi) = 0$$

if $\pi \notin \mathscr{R}(MO(p,q), \omega(p,q;2n))$. We define $\theta(p,q;2n)(0) = 0$ and 0 can be regarded as the NULL representation.

For example, given an "increasing" string

$$O(p_1,q_1) \to Sp_{2n_1}(\mathbb{R}) \to O(p_2,q_2) \to Sp_{2n_2}(\mathbb{R}) \to \cdots \to Sp_{2n_m}(\mathbb{R}) \to O(p_m,q_m),$$

$$p_1 + q_1 \equiv p_2 + q_2 \equiv \cdots \equiv p_m + q_m \pmod 2,$$

consider the propagation of theta correspondence along this string:

$$\theta(2n_m; p_m, q_m) \ldots \theta(2n_1; p_2, q_2)\theta(p_1, q_1; 2n_1)(\pi).$$