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Enumeration of totally positive Grassmann cells

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Abstract

Postnikov (Webs in totally positive Grassmann cells, in preparation) has given a combinatorially explicit cell decomposition of the totally nonnegative part of a Grassmannian, denoted $Gr^+_{k,n}$, and showed that this set of cells is isomorphic as a graded poset to many other interesting graded posets. The main result of our work is an explicit generating function which enumerates the cells in $Gr^+_{k,n}$ according to their dimension. As a corollary, we give a new proof that the Euler characteristic of $Gr^+_{k,n}$ is 1. Additionally, we use our result to produce a new q-analog of the Eulerian numbers, which interpolates between the Eulerian numbers, the Narayana numbers, and the binomial coefficients.

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1. Introduction

The classical theory of total positivity concerns matrices in which all minors are nonnegative. While this theory was pioneered by Gantmacher, Krein, and Schoenberg in the 1930s, the past decade has seen a flurry of research in this area initiated by Lusztig [4–6]. Motivated by surprising connections he discovered between his theory of canonical bases for quantum groups and the theory of total positivity, Lusztig extended this subject by introducing the totally nonnegative variety $G_{\geqslant 0}$ in an arbitrary reductive group G and the totally nonnegative part $B_{\geqslant 0}$

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of a real flag variety B. A few years later, Fomin and Zelevinsky [2] advanced the understanding of $G_{\geqslant 0}$ by studying the decomposition of G into double Bruhat cells, and Rietsch [8] proved Lusztig's conjectural cell decomposition of $B_{\geqslant 0}$. Most recently, Postnikov [7] investigated the combinatorics of the totally nonnegative part of a Grassmannian $Gr^+_{k,n}$: he established a relationship between $Gr^+_{k,n}$ and planar oriented networks, producing a combinatorially explicit cell decomposition of $Gr^+_{k,n}$. In this paper we continue Postnikov's study of the combinatorics of $Gr^+_{k,n}$: in particular, we enumerate the cells in the cell decomposition of $Gr^+_{k,n}$ according to their dimension.

The *totally nonnegative part* of the Grassmannian of k-dimensional subspaces in $\mathbb{R}^n$ is defined as $Gr^+_{k,n} = \mathrm{GL}^+_k \backslash \mathrm{Mat}^+(k,n)$, where $\mathrm{Mat}^+(k,n)$ is the space of real $k \times n$-matrices of rank k with nonnegative maximal minors and GL^+_k is the group of real matrices with positive determinant. If we specify which maximal minors are strictly positive and which are equal to zero, we obtain a cellular decomposition of $Gr^+_{k,n}$, as shown in [7]. We refer to the cells in this decomposition as *totally positive cells*. The set of totally positive cells naturally has the structure of a graded poset: we say that one cell covers another if the closure of the first cell contains the second, and the rank function is the dimension of each cell.

Lusztig [6] has proved that the totally nonnegative part of the (full) flag variety is contractible, which implies the same result for any partial flag variety. (We thank K. Rietsch for pointing this out to us.) The topology of the individual cells is not well understood, however. Postnikov [7] has conjectured that the closure of each cell in $Gr^+_{k,n}$ is homeomorphic to a closed ball.

In [7], Postnikov constructed many different combinatorial objects which are in one-to-one correspondence with the totally positive Grassmann cells (these objects thereby inherit the structure of a graded poset). Some of these objects include decorated permutations, Ⅎ-diagrams, positive oriented matroids, and move-equivalence classes of planar oriented networks. Because it is simple to compute the rank of a particular Ⅎ-diagram or decorated permutation, we will restrict our attention to these two classes of objects.

The main result of this paper is an explicit formula for the *rank generating function* $A_{k,n}(q)$ of $Gr^+_{k,n}$. Specifically, $A_{k,n}(q)$ is defined to be the polynomial in q whose q^r coefficient is the number of totally positive cells in $Gr^+_{k,n}$ which have dimension r. As a corollary of our main result, we give a new proof that the Euler characteristic of $Gr^+_{k,n}$ is 1. Additionally, using our result and exploiting the connection between totally positive cells and permutations, we compute generating functions which enumerate (regular) permutations according to two statistics. This leads to a new q-analog of the Eulerian numbers that has many interesting combinatorial properties. For example, when we evaluate this q-analog at $q = 1, 0,$ and -1, we obtain the Eulerian numbers, the Narayana numbers, and the binomial coefficients. Finally, the connection with the Narayana numbers suggests a way of incorporating noncrossing partitions into a larger family of "crossing" partitions.