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The compactified Picard scheme of the compactified Jacobian

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Abstract

Let C be an integral projective curve in any characteristic. Given an invertible sheaf $\mathcal{L}$ on C of degree 1, form the corresponding Abel map $A_{\mathcal{L}}: C \to \bar{J}$, which maps C into its compactified Jacobian, and form its pullback map $A^*_{\mathcal{L}}: \mathrm{Pic}^0_{\bar{J}} \to J$, which carries the connected component of 0 in the Picard scheme back to the Jacobian. If C has, at worst, double points, then $A^*_{\mathcal{L}}$ is known to be an isomorphism. We prove that $A^*_{\mathcal{L}}$ always extends to a map between the natural compactifications, $\mathrm{Pic}^{\div}_{\bar{J}} \to \bar{J}$, and that the extended map is an isomorphism if C has, at worst, ordinary nodes and cusps.

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1. Introduction

Let C be an integral projective curve of arithmetic genus g, defined over an algebraically closed field of any characteristic. Form its (generalized) Jacobian J, the connected component of the identity of the Picard scheme of C. If C is singular, then J is not projective. So for about forty years, numerous authors have studied a natural compactification of J: the (fine) moduli space $\bar{J}$ of torsion-free sheaves of rank 1 and degree 0 on C. It is called the *compactified Jacobian*.

Recently, the compactified Jacobian appeared in Laumon's paper [10], where he identified, up to homeomorphism, affine Springer fibers with coverings of compactified Jacobians. For that identification, he used the autoduality of the compactified Jacobian, a property established in [6] and explained next.

From now on, assume C has, at worst, points of multiplicity 2 (or double points). For each invertible sheaf $\mathcal{L}$ of degree 1 on C, form the Abel map $A_{\mathcal{L}}: C \to \bar{J}$, given by $P \mapsto \mathcal{M}_P \otimes \mathcal{L}$ where $\mathcal{M}_P$ is the ideal sheaf of P; it is a closed embedding if C is not of genus 0. Form the pullback map

$$A_{\mathcal{L}}^*: \mathrm{Pic}^0_{\bar{J}} \to J,$$

carrying the connected component of 0 in the Picard scheme back to the Jacobian. Then $A_{\mathcal{L}}^*$ is an isomorphism and is independent of $\mathcal{L}$; see [6, Theorem 2.1, p. 595].

Since the singularities are locally planar, $\bar{J}$ is integral by [1, (9), p. 8]. Hence, not only does $\mathrm{Pic}^0_{\bar{J}}$ exist, but also it admits a natural compactification: its closure $\mathrm{Pic}^{\div}_{\bar{J}}$ in the compactified Picard scheme $\mathrm{Pic}^{=}_{\bar{J}}$, the (fine) moduli space of torsion-free sheaves of rank 1 on $\bar{J}$; see [3, Theorem 3.1, p. 28]. Does $A_{\mathcal{L}}^*$ extend to a map between the compactifications? If so, then is the extension an isomorphism?

These questions were posed to the authors by Sawon. As mentioned in his introduction to [11], his results on dual fibrations to fibrations by Abelian varieties, in the "nicest" cases, depend on "extending autoduality to the compactifications."

It is not true, for every map, that the pullback of a torsion-free sheaf is still torsion free. But, for $A_{\mathcal{L}}$, it is true! There are two basic reasons why: first, $A_{\mathcal{L}}^*$ is independent of $\mathcal{L}$; second, the maps $A_{\mathcal{L}}$ can be bundled up into a *smooth* map $C \times J \to \bar{J}$. Thus there exists an extended pullback map

$$A_{\mathcal{L}}^*: \mathrm{Pic}^{\div}_{\bar{J}} \to \bar{J};$$

this existence statement is the content of Theorem 2.6 below. (The statement and its proof already appear on the web in the preliminary version of [6].)

Is the extended map $A_{\mathcal{L}}^*$ also an isomorphism? This question seems much harder.

From now on, assume that C has, at worst, ordinary nodes and cusps. Then the extended map $A_{\mathcal{L}}^*$ is, indeed, an isomorphism, according to Theorem 4.1, our main result. Here is a sketch of the proof.

First, recall the definition of the inverse $\beta: J \to \mathrm{Pic}^0_{\bar{J}}$ from [6, Proposition 2.2, p. 595], or rather [6, Remark 2.4, p. 597]. Let $\mathcal{I}$ be the universal sheaf on $C \times \bar{J}$, and $\mathcal{P}$