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Quantum cluster algebras☆

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Abstract

Cluster algebras form an axiomatically defined class of commutative rings designed to serve as an algebraic framework for the theory of total positivity and canonical bases in semisimple groups and their quantum analogs. In this paper we introduce and study quantum deformations of cluster algebras.

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1. Introduction

Cluster algebras were introduced by S. Fomin and A. Zelevinsky [8]; their study continued in [10,2]. This is a family of commutative rings designed to serve as an algebraic framework for the theory of total positivity and canonical bases in semisimple groups and their quantum analogs. In this paper, we introduce and study quantum deformations of cluster algebras.

Our immediate motivation for introducing quantum cluster algebras is to prepare the ground for a general notion of the canonical basis in a cluster algebra. Remarkably, cluster algebras and their quantizations appear to be relevant for the study of (higher)

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Teichmuller theory initiated in [11,12,5,6]. Our approach to quantization has much in common with the one in [5,6], but we develop it more systematically. In particular, we show that practically all the structural results on cluster algebras obtained in [8,10,2] extend to the quantum setting. This includes the Laurent phenomenon [8,9,2] and the classification of cluster algebras of finite type [10].

Our approach to quantum cluster algebras can be described as follows. Recall that a *cluster algebra* $\mathcal{A}$ is a certain commutative ring generated by a (possibly infinite) set of generators called *cluster variables* inside an ambient field $\mathcal{F}$ isomorphic to the field of rational functions in m independent variables over $\mathbb{Q}$. The set of cluster variables is the union of some distinguished transcendence bases of $\mathcal{F}$ called (extended) *clusters*. The clusters are not given from the outset but are obtained from an initial cluster via an iterative process of *mutations* which follows a set of canonical rules. According to these rules, every cluster $\{x_1, \ldots, x_m\}$ is surrounded by n adjacent clusters (for some $n \leqslant m$ called the *rank* of $\mathcal{A}$) of the form $\{x_1, \ldots, x_m\} - \{x_k\} \cup \{x'_k\}$, where k runs over a given n-element subset of *exchangeable* indices, and $x'_k \in \mathcal{F}$ is related to x_k by the *exchange relation* (see (2.2)). The cluster algebra structure is completely determined by an $m \times n$ integer matrix $\tilde{B}$ that encodes all the exchange relations. (The precise definitions of all these notions are given in Section 2.) Now, the quantum deformation of $\mathcal{A}$ is a $\mathbb{Q}(q)$-algebra obtained by making each cluster into a *quasi-commuting* family $\{X_1, \ldots, X_m\}$; this means that $X_i X_j = q^{\lambda_{ij}} X_j X_i$ for a skew-symmetric integer $m \times m$ matrix $\Lambda = (\lambda_{ij})$. In doing so, we have to modify the mutation process and the exchange relations so that all the adjacent quantum clusters will also be quasi-commuting. This imposes the *compatibility* relation between the quasi-commutation matrix Λ and the exchange matrix $\tilde{B}$ (Definition 3.1). In what follows, we develop a formalism that allows us to show that any compatible matrix pair $(\Lambda, \tilde{B})$ gives rise to a well-defined quantum cluster algebra.

The paper is organized as follows. In Section 2, we present necessary definitions and facts from the theory of cluster algebras in the form suitable for our current purposes. In Section 3, we introduce compatible matrix pairs $(\Lambda, \tilde{B})$ and their mutations.

Section 4 plays the central part in this paper. It introduces the main concepts needed for the definition of quantum cluster algebras (Definition 4.12): *based quantum tori* (Definition 4.1) and their skew-fields of fractions, *toric frames* (Definition 4.3), *quantum seeds* (Definition 4.5) and their mutations (Definition 4.8).

Section 5 establishes the quantum version of the Laurent phenomenon (Corollary 5.2): any cluster variable is a Laurent polynomial in the elements of any given cluster. The proof closely follows the argument in [2] with necessary modifications. It is based on the important concept of an *upper cluster algebra* and the fact that it is invariant under mutations (Theorem 5.1).

In Section 6, we show that the *exchange graph* of a quantum cluster algebra remains unchanged in the “classical limit” $q = 1$ (Theorem 6.1). (Recall that the vertices of the exchange graph correspond to (quantum) seeds, and the edges correspond to mutations.) An important consequence of Theorem 6.1 is that the classification of cluster algebras of finite type achieved in [10] applies verbatim to quantum cluster algebras.

An important ingredient of the proof of Theorem 6.1 is the *bar-involution* on the quantum cluster algebra which is modeled on the Kazhdan–Lusztig involution, or the