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Smallest singular value of random matrices and geometry of random polytopes

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Abstract

We study the behaviour of the smallest singular value of a rectangular random matrix, i.e., matrix whose entries are independent random variables satisfying some additional conditions. We prove a deviation inequality and show that such a matrix is a "good" isomorphism on its image. Then, we obtain asymptotically sharp estimates for volumes and other geometric parameters of random polytopes (absolutely convex hulls of rows of random matrices). All our results hold with high probability, that is, with probability exponentially (in dimension) close to 1.

Keywords: Random matrices; Random polytopes; Singular values; Deviation inequalities

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1. Introduction

In this paper, we consider rectangular $N \times n$ random matrices, whose entries are independent and satisfy some moment conditions, and such that the whole matrix satisfies additional boundedness conditions. We are interested in singular values of such matrices and in geometric parameters of the polytopes they determine.

Assume that $N \geqslant n$ and denote such a matrix by $\Gamma = [\xi_{ij}]_{1 \leqslant i \leqslant N, 1 \leqslant j \leqslant n}$. Let us briefly recall some known results on singular values of Γ. Assume that the variance of the ξ_{ij} is 1, and that N is proportional to n, say $n/N = c$ (where c is considered fixed). From a result in [MP], the empirical measure associated to the spectrum of the sample covariance matrix $\Gamma^*\Gamma/N$ has a deterministic limit distribution supported by the interval $[(1-\sqrt{c})^2, (1+\sqrt{c})^2]$. More precisely, by results from [Si] in the Gaussian case, and from [BY] in the general case (assuming the finite fourth moment of the ξ_{ij}'s), we get that the smallest eigenvalue converges a.e. to $(1-\sqrt{c})^2$. Let $s_n = s_n(\Gamma)$ be the smallest singular value of Γ. Then the above statement says, after a renormalization, that $s_n/\sqrt{N} \to 1 - \sqrt{c}$ a.e., as $N \to \infty$. However, the concentration of this random variable around $1 - \sqrt{c}$ is in general unknown.

In this paper, we give an (exponentially small) upper estimate for the probability that $s_n/\sqrt{N}$ is small. Denoting by $\|\cdot\|$ the operator norm of an operator acting on a Hilbert space, and considering Γ as acting onto its image, we show (in Theorem 3.1) that for any $0 < c < 1$ there is a function $\phi(c)$ such that the embedding Γ satisfies $\|\Gamma\| \, \|\Gamma^{-1}\| \leqslant \phi(c)$, for any N and n such that $n/N \leqslant c$, with probability larger than $1 - \exp(-c_2 N)$, for some fixed $c_2 > 0$. To the contrary to the approach discussed above, when the ratio $c = n/N$ is considered fixed (independent of n and N), in the present paper we consider n and N to be independent parameters, in particular, allowing c to depend on n. This result can be interpreted by saying that if $n/N \leqslant c$ then, with high probability, Γ is a "good" isomorphic embedding of ℓ_2^n into ℓ_2^N (Let us also mention that in [LPRTV] a similar result is proved for embeddings of ℓ_2^n into a large class of spaces which, for example, includes ℓ_1^N.)

Theorem 3.1 is then applied to study geometry of random polytopes generated by Γ, that is, the absolute convex hull of N rows of Γ. Such random polytopes have been extensively studied in the Gaussian case, as well as the Bernoulli case. The former case, when N is proportional to n, has many applications in the asymptotic theory of normed spaces (see e.g., [G1,Sz], and the survey [MT]). In the Bernoulli case, random polytopes of this form have been investigated in [GH], as well as in a combinatorial setting of the so-called 0-1 polytopes (see for instance [DFM,BP], and the survey [Z]).

When speaking of random matrices, we identify a large class that contains the most important cases studied in the literature, such as the case when the entries are Gaussian or Bernoulli random variables.

Let us now briefly describe the organization of the paper. In Section 2, we introduce the class of matrices that we consider and we prove some basic facts about them. In Section 3, we show, in Theorem 3.1, that if n is arbitrary and $N = (1+\delta)n$ (where