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Comparing the uniformity invariants of null sets for different measures

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Abstract

It is shown to be consistent with set theory that the uniformity invariant for Lebesgue measure is strictly greater than the corresponding invariant for Hausdorff r-dimensional measure where $0<r<1$.

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1. Introduction

The uniformity invariant for Lebesgue measure is defined to be the least cardinal of a non-measurable set of reals, or, equivalently, the least cardinal of a set of reals

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which is not Lebesgue null. This has been studied intensively for the past 30 years and much of what is known can be found in [1] and other standard sources. Among the well known results about this cardinal invariant of the continuum is that it can equally well be defined using Lebesgue measure on $\mathbb{R}^n$ without changing the value of the cardinal. Indeed, equivalent definitions will result by using any Borel probability measure on any Polish space. However, the question of the values of uniformity invariants for other, non-σ-finite Borel measures is not so easily answered. This paper will deal with the most familiar class of such measures, the Hausdorff measures for fractional dimension. Observe that by the previous remarks, the least cardinal of any non-measurable subset of any σ-finite set will be the same as the uniformity invariant for Lebesgue measure. In other words, this paper will be concerned with the uniformity invariant of the ideal of σ-finite sets with respect to a Hausdorff measure.

It will be shown that given any real number r in the interval $(0, 1)$ it is consistent with set theory that every set of reals of size $\aleph_1$ is Lebesgue measurable yet there is a set of reals of size $\aleph_1$ which is not a null set with respect to r-dimensional Hausdorff measure. This answers Question FQ from D. Fremlin's list of open questions [2]. However, the motivation was an attempt to resolve the following question posed by P. Komjáth.

Question 1.1. Suppose that every set of size $\aleph_1$ has Lebesgue measure zero. Does it follow that the union of any set of $\aleph_1$ lines in the plane has Lebesgue measure zero?

It is worth noting that this is really a geometric question since [9] provides a negative answer to the version of the problem in which lines are replaced by their topological and measure theoretic equivalents. To see the relationship between Question 1.1 and the topic of this paper consider that it is easy to find countably many unit squares in the plane such that each line passes through either the top and bottom or the left and right sides of at least one of these squares. It is therefore possible to focus attention on all lines which pass through the top and bottom of the unit square. For any such line L there is a pair (a, b) such that both the points $(a, 0)$ and $(1, b)$ belong to L. If the mapping which sends a line L to this pair (a, b) is denoted by β then it is easy to see that β is continuous and that if $S \subseteq [0, 1]^2$ is a square of side ε then the union of $\beta^{-1}S$ has measure ε while S itself has measure ε^2. In other words, the Lebesgue measure of the union of the lines belonging to $\beta^{-1}X$ is no larger than the one-dimensional Hausdorff measure of X for any $X \subseteq [0, 1]^2$. In other words, a negative answer to Question 1.1 would imply that there is $X \subseteq [0, 1]^2$ of size $\aleph_1$ which is not null with respect to linear Hausdorff measure even though every set of reals of size $\aleph_1$ is null. The consistency of this will be a consequence of Corollary 6.2.

The proof will rely partially on arguments from [7,8] in which a single stage of a forcing iteration that would achieve the desired model was described. The material in Sections 3 and 4 is a reorganized and simplified version of Sections 4–6 of [8] which has been suitably modified for the current context. The approach taken here differs