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Structure of the Malvenuto–Reutenauer Hopf algebra of permutations

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Abstract

We analyze the structure of the Malvenuto–Reutenauer Hopf algebra $\mathfrak{S}$Sym of permutations in detail. We give explicit formulas for its antipode, prove that it is a cofree coalgebra, determine its primitive elements and its coradical filtration, and show that it decomposes as a crossed product over the Hopf algebra of quasi-symmetric functions. In addition, we describe the structure constants of the multiplication as a certain number of facets of the permutahedron. As a consequence we obtain a new interpretation of the product of monomial quasi-symmetric functions in terms of the facial structure of the cube. The Hopf algebra of Malvenuto and Reutenauer has a linear basis indexed by permutations. Our results are obtained from a combinatorial description of the Hopf algebraic structure with respect to a new basis for this algebra, related to the original one via Möbius inversion on the weak order on the symmetric groups. This is in analogy with the relationship between the monomial and fundamental bases of the algebra of quasi-symmetric functions. Our results reveal a close relationship between the structure of the Malvenuto–Reutenauer Hopf algebra and the weak order on the symmetric groups.

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Introduction

Malvenuto [22] introduced the Hopf algebra $\mathfrak{S}$Sym of permutations, which has a linear basis $\{\mathcal{F}_u \mid u \in \mathfrak{S}_n, n \geqslant 0\}$ indexed by permutations in all symmetric groups $\mathfrak{S}_n$. The Hopf algebra $\mathfrak{S}$Sym is non-commutative, non-cocommutative, self-dual, and graded. Among its sub-, quotient-, and subquotient-Hopf algebras are many algebras central to algebraic combinatorics. These include the algebra of symmetric functions [21,33], Gessel's algebra $\mathcal{Q}$Sym of quasi-symmetric functions [13], the algebra of non-commutative symmetric functions [12], the Loday–Ronco algebra of planar binary trees [19], Stembridge's algebra of peaks [34], the Billera–Liu algebra of Eulerian enumeration [2], and others. The structure of these combinatorial Hopf algebras with respect to certain distinguished bases has been an important theme in algebraic combinatorics, with applications to the combinatorial problems these algebras were created to study. Here, we obtain a detailed understanding of the structure of $\mathfrak{S}$Sym, both in algebraic and combinatorial terms.

Our main tool is a new basis $\{\mathcal{M}_u \mid u \in \mathfrak{S}_n,\ n \geqslant 0\}$ for $\mathfrak{S}$Sym related to the original basis by Möbius inversion on the weak order on the symmetric groups. These bases $\{\mathcal{M}_u\}$ and $\{\mathcal{F}_u\}$ are analogous to the monomial basis and the fundamental basis of $\mathcal{Q}$Sym, which are related via Möbius inversion on their index sets, the Boolean posets $\mathcal{Q}_n$. We refer to them as the monomial basis and the fundamental basis of $\mathfrak{S}$Sym.

We give enumerative-combinatorial descriptions of the product, coproduct, and antipode of $\mathfrak{S}$Sym with respect to the monomial basis $\{\mathcal{M}_u\}$. In Section 3, we show that the coproduct is obtained by splitting a permutation at certain special positions that we call global descents. Descents and global descents are left adjoint and right adjoint to a natural map $\mathcal{Q}_n \to \mathfrak{S}_n$. These results rely on some non-trivial properties of the weak order developed in Section 2.

The product is studied in Section 4. The structure constants are non-negative integers with the following geometric-combinatorial description. The 1-skeleton of the permutahedron Π_{n-1} is the Hasse diagram of the weak order on $\mathfrak{S}_n$. The facets of the permutahedron are canonically isomorphic to products of lower dimensional permutahedra. Say that a facet isomorphic to $\Pi_{p-1} \times \Pi_{q-1}$ has type (p,q). Given $u \in \mathfrak{S}_p$ and $v \in \mathfrak{S}_q$, such a facet has a distinguished vertex corresponding to (u,v) under the canonical isomorphism. Then, for $w \in \mathfrak{S}_{p+q}$, the coefficient of $\mathcal{M}_w$ in $\mathcal{M}_u \cdot \mathcal{M}_v$ is the number of facets of the permutahedron Π_{p+q-1} of type (p,q) with the property that the distinguished vertex is below w (in the weak order) and closer to w than any other vertex in the facet.

In Section 5 we find explicit formulas for the antipode with respect to both bases. The structure constants with respect to the monomial basis have constant sign, as for $\mathcal{Q}$Sym. The situation is more complicated for the fundamental basis, which may explain why no such explicit formulas were previously known.

Elucidating the elementary structure of $\mathfrak{S}$Sym with respect to the monomial basis reveals further algebraic structures of $\mathfrak{S}$Sym. In Section 6, we show that $\mathfrak{S}$Sym is a cofree graded coalgebra. A consequence is that its coradical filtration (a filtration encapsulating the complexity of iterated coproducts) is the algebraic counterpart of a