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# The homotopy category of complexes of projective modules

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**Abstract**

The homotopy category of complexes of projective left-modules over any reasonably nice ring is proved to be a compactly generated triangulated category, and a duality is given between its subcategory of compact objects and the finite derived category of right-modules.


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*Keywords:* Coherent ring; Flat module; Projective dimension; Compactly generated triangulated category; Derived category; Dualizing complex

## 0. Introduction

The last decade has seen compactly generated triangulated categories rise to prominence. Triangulated categories go back to Puppe and Verdier, but only later developments have made it clear that the compactly generated ones are particularly useful. For instance, they allow the use of the Brown Representability Theorem and the Thomason Localization Theorem, both proved by Neeman in [6]. There are also results by many other authors to support the case.

The standard examples of compactly generated triangulated categories are the stable homotopy category of spectra and the derived category of a ring. Indeed, many analogies between these two cases are captured by their common structure of compactly generated

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triangulated category, and this allows the transfer of methods and ideas back and forth.

This paper adds to the collection of compactly generated triangulated categories by showing that if $A$ is a reasonably nice ring, then the homotopy category of complexes of projective $A$-left-modules, $\mathsf{K}(\mathsf{Pro}\, A)$, is compactly generated. Furthermore, the subcategory of compact objects, $\mathsf{K}(\mathsf{Pro}\, A)^{\mathsf{c}}$, admits a duality of categories

$$\mathsf{K}(\mathsf{Pro}\, A)^{\mathsf{c}} \leftrightarrow \mathsf{D}^{\mathsf{f}}(A^{\mathrm{op}}), \tag{1}$$

where $\mathsf{D}^{\mathsf{f}}(A^{\mathrm{op}})$ is the finite derived category of $A$-right-modules whose objects are complexes with bounded cohomology consisting of finitely presented modules.

Moreover, if $A$ is a reasonably nice $k$-algebra over the field $k$, and if there exists a $k$-algebra $B$ and a dualizing complex ${}_B D_A$, as defined in [8, Definition 1.1], then the duality of categories can be improved to an equivalence

$$\mathsf{K}(\mathsf{Pro}\, A)^{\mathsf{c}} \xrightarrow{\simeq} \mathsf{D}^{\mathsf{f}}(B). \tag{2}$$

My proof that $\mathsf{K}(\mathsf{Pro}\, A)$ is compactly generated and admits the duality (1) works when $A$ is a coherent ring for which each flat left-module has finite projective dimension, and my proof of the equivalence (2) works when $A$ is a left-coherent and right-noetherian $k$-algebra which admits a left-noetherian $k$-algebra $B$ and a dualizing complex ${}_B D_A$.

This covers a wide variety of natural examples. For instance, many rings, such as noetherian rings, are coherent. The condition that each flat $A$-left-module has finite projective dimension would appear less standard, but is in fact satisfied by large classes of rings such as noetherian commutative rings of finite Krull dimension [7, Seconde partie, cor. (3.2.7)], left-perfect rings [1, Theorem P], and right-noetherian algebras which admit a dualizing complex [3].

It is worth noting that if $A$ has finite left and right global dimension, then there is nothing new in my results. In this case, [4, Lemma 1.7] gives that $\mathsf{K}(\mathsf{Pro}\, A)$ is equivalent to $\mathsf{D}(A)$, the derived category of $A$-left-modules, so $\mathsf{K}(\mathsf{Pro}\, A)$ is compactly generated. ([4, Lemma 1.7] is formulated for $\mathsf{K}(\mathsf{Free}\, A)$, but removing part of the proof gives an argument that works for $\mathsf{K}(\mathsf{Pro}\, A)$.) The subcategory of compact objects $\mathsf{K}(\mathsf{Pro}\, A)^{\mathsf{c}}$ is now clearly equivalent to $\mathsf{D}(A)^{\mathsf{c}}$, the subcategory of compact objects of $\mathsf{D}(A)$. It follows from [6, Theorem 2.1.3] that $\mathsf{D}(A)^{\mathsf{c}}$ consists of the complexes which are isomorphic to bounded complexes of finitely generated projective modules. The same holds for $\mathsf{D}(A^{\mathrm{op}})^{\mathsf{c}}$, and therefore $\mathrm{RHom}_A(-, A)$ induces a duality between $\mathsf{D}(A)^{\mathsf{c}}$ and $\mathsf{D}(A^{\mathrm{op}})^{\mathsf{c}}$. And finally, since $A$ has finite right global dimension, each complex in $\mathsf{D}^{\mathsf{f}}(A^{\mathrm{op}})$ has a bounded resolution consisting of finitely generated projective modules, so $\mathsf{D}^{\mathsf{f}}(A^{\mathrm{op}})$ is equal to $\mathsf{D}(A^{\mathrm{op}})^{\mathsf{c}}$.

These standard arguments show that if $A$ has finite left and right global dimension, then $\mathsf{K}(\mathsf{Pro}\, A)$ is compactly generated and admits the duality of categories in Eq. (1). However, my results show the same for many rings which do not have finite global dimension.