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Orthonormal expansions of invariant densities for expanding maps

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Abstract

We give a novel way of constructing the density function for the absolutely continuous invariant measure of piecewise expanding C^ω Markov maps. This is a classical problem, with one of the standard approaches being Ulam's method [Problems in Modern Mathematics, Interscience, New York, 1960] of phase space discretisation.

Our method hinges instead on the expansion of the density function with respect to an L^2 orthonormal basis, and the computation of the expansion coefficients in terms of the periodic orbits of the expanding map. The efficiency of the method, and its extension to C^k expanding maps, are also discussed.

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1. Introduction

Let T be either a C^ω (i.e. real analytic) expanding map of the circle, or a piecewise C^ω expanding Markov map (see Section 2 for the definition) of the interval. It is well known that there exists a T-invariant probability measure μ which is absolutely

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continuous with respect to Lebesgue measure (cf. [LY]). If T is topologically mixing[1] then this absolutely continuous invariant measure is unique and ergodic, with strictly positive C^ω density function ϱ.

A central problem in ergodic theory is the computation of time averages of the form $\lim_{n\to\infty} 1/n \sum_{r=0}^{n-1} g(T^r x)$, for integrable functions g. The straightforward evaluation of $1/N \sum_{r=0}^{N-1} g(T^r x)$ for large N is known to be an inefficient way of approximating the limit (cf. [ER]). An application of Birkhoff's ergodic theorem reduces the problem, for Lebesgue almost every x, to a computation of the space average $\int g\, d\mu = \int g(x)\varrho(x)\, dx$. However for general non-linear expanding maps the density ϱ is not explicitly known (i.e. is not available in closed form), so that an approximation scheme for evaluating $\int g\, d\mu$ is required.

In [JP] we gave an efficient method for computing such integrals, provided the function g is real analytic. The periodic points of period up to M are used to construct a sequence of approximations to $\int g\, d\mu$, whose rate of convergence to $\int g\, d\mu$ is *superexponential in* M. Thus we have a very rapidly convergent algorithm for computing the time average of a given (real analytic) function g.

If, however, we want to compute time averages for *many* functions g, the required high number of iterations of our algorithm may outweigh its inherent efficiency. In this case it would be preferable to accurately approximate, once and for all, the density function ϱ. Thereafter, for all functions g, which now may be chosen as merely L^1 rather than C^ω, the integral $\int g\, d\mu = \int g(x)\varrho(x)\, dx$ can be approximated *directly*, without recourse to an algorithm. That is, each integral may be evaluated approximately by either a symbolic or a numerical method, and the time required for such evaluations is comparatively very small. The classical method for approximating density functions, first proposed by Ulam [U], is to discretise the system by partitioning the phase space. The density is then approximated by the dominant eigenvector of a finite matrix. Recent work on Ulam's method includes [DS,Froy,Hunt,KMY,Liv,Mu].

In the present paper we will describe how to perform a one-off computation of ϱ using periodic points, and give error estimates in terms of the largest period used. The starting point is to view the analytic function ϱ as an element of a suitable L^2 space, and expand $\varrho = \sum_n (\varrho, f_n) f_n$ in terms of an orthonormal basis $\{f_n\}$. If the interval is parametrised as $I = [-1, 1]$, then a natural expansion is in terms of normalised *Legendre polynomials* $f_n = (n + \frac{1}{2})^{1/2} P_n$, for n, where P_n is the nth Legendre polynomial (see Section 3 for the definition). In this case

$$(\varrho, f_n) = \int_{-1}^{1} \varrho(x)\overline{f_n(x)}\, dx = \int \overline{f_n}\, d\mu$$

is the inner product in the Hilbert space $L^2(I, dx)$.

If T is a map of the circle then the density ϱ will be analytic, and hence L^2, on the circle. Equivalently, ϱ can be considered as a continuous function on the

[1] Mixing is automatic if T is an expanding circle map.