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U(1)-invariant special Lagrangian 3-folds. III. Properties of singular solutions

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Abstract

This is the last of three papers studying *special Lagrangian* 3-*submanifolds* (*SLV* 3-*folds*) N in $\mathbb{C}^3$ invariant under the U(1)-action $e^{i\theta}:(z_1, z_2, z_3) \mapsto (e^{i\theta} z_1, e^{i\theta} z_2, z_3)$, using analytic methods. If N is such a 3-fold then $|z_1|^2 - |z_2|^2 = 2a$ on N for some $a \in \mathbb{R}$. Locally, N can be written as a kind of graph of functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ satisfying a *nonlinear Cauchy–Riemann equation* depending on a, so that $u + iv$ is like a holomorphic function of $x + iy$.

The first paper studied the case a nonzero, and proved existence and uniqueness for solutions of two *Dirichlet problems* derived from the nonlinear Cauchy–Riemann equation. This yields existence and uniqueness of a large class of *nonsingular* U(1)-invariant SL 3-folds in $\mathbb{C}^3$, with boundary conditions. The second paper extended these results to *weak solutions* of the Dirichlet problems when $a = 0$, giving existence and uniqueness of many *singular* U(1)-invariant SL 3-folds in $\mathbb{C}^3$, with boundary conditions.

This third paper studies the *singularities* of these SL 3-folds. We show that under mild conditions the singularities are *isolated*, and have a *multiplicity* $n > 0$, and one of two *types*. Examples are constructed with every multiplicity and type. We also prove the existence of large families of U(1)-invariant *special Lagrangian fibrations* of open sets in $\mathbb{C}^3$, including singular fibres.

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1. Introduction

Special Lagrangian submanifolds (SL m-folds) are a distinguished class of real m-dimensional minimal submanifolds in $\mathbb{C}^m$, which are calibrated with respect to the m-form $\mathrm{Re}(\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m)$. They can also be defined in Calabi–Yau manifolds, are important in String Theory, and are expected to play a rôle in the eventual explanation of Mirror Symmetry between Calabi–Yau 3-folds.

This is the third in a suite of three papers [9,10] studying special Lagrangian 3-folds N in $\mathbb{C}^3$ invariant under the U(1)-action

$$\mathrm{e}^{i\theta} : (z_1, z_2, z_3) \mapsto (\mathrm{e}^{i\theta} z_1, \mathrm{e}^{-i\theta} z_2, z_3) \quad \text{for } \mathrm{e}^{i\theta} \in \mathrm{U}(1). \tag{1}$$

These three papers and [11] are reviewed in [12]. Locally, we can write N as

$$N = \Big\{ (z_1, z_2, z_3) \in \mathbb{C}^3 : \mathrm{Im}(z_3) = u(\mathrm{Re}(z_3), \mathrm{Im}(z_1 z_2)), \\ \mathrm{Re}(z_1 z_2) = v(\mathrm{Re}(z_3), \mathrm{Im}(z_1 z_2)), \quad |z_1|^2 - |z_2|^2 = 2a \Big\}, \tag{2}$$

where $a \in \mathbb{R}$ and $u, v : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions. It was shown in [9] that when $a \neq 0$, N is an SL 3-fold in $\mathbb{C}^3$ if and only if u, v satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -2(v^2 + y^2 + a^2)^{1/2} \frac{\partial u}{\partial y}, \tag{3}$$

and then u, v are smooth and N is nonsingular.

Here is what we mean by saying N is locally of the form (2). *Every* connected U(1)-invariant SL 3-fold N may be written in the form (2) for u, v continuous, *multi-valued* 'functions' (multifunctions). Since (3) is a *nonlinear Cauchy–Riemann equation*, $u + iv$ is a bit like a holomorphic function of $x + iy$, and so the multifunctions (u, v) behave like holomorphic multifunctions in complex analysis, such as $\sqrt{z}$.

Thus we expect them to have isolated *branch points* in $\mathbb{R}^2$. Over simply connected open sets $U \subset \mathbb{R}^2$ not containing any branch points, the multifunctions (u, v) decompose into *sheets*, each of which is a continuous, single-valued function. This is like choosing a branch of $\sqrt{z}$ on a simply connected open subset $U \subset \mathbb{C} \setminus \{0\}$. In terms of N, we expect there to be a discrete set of 'branch point' U(1)-orbits, and small U(1)-invariant open sets in N away from these can be written in the form (2) for single-valued (u, v).

As $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ there exists $f : \mathbb{R}^2 \to \mathbb{R}$ with $\frac{\partial f}{\partial y} = u$ and $\frac{\partial f}{\partial x} = v$, satisfying

$$\left(\left(\frac{\partial f}{\partial x} \right)^2 + y^2 + a^2 \right)^{-1/2} \frac{\partial^2 f}{\partial x^2} + 2 \frac{\partial^2 f}{\partial y^2} = 0. \tag{4}$$

In [9, Theorem 7.6] we proved existence and uniqueness for the Dirichlet problem for (4) in strictly convex domains S in $\mathbb{R}^2$ when $a \neq 0$. This gives existence and uniqueness of a large class of nonsingular U(1)-invariant SL 3-folds in $\mathbb{C}^3$ satisfying certain boundary conditions.