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Geometry of formal Kuranishi theory

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Abstract

The principle "ambient cohomology of a Kaehler manifold annihilates obstructions" has been known and exploited since pioneering work of Kodaira. This paper unifies and modestly extends known results in the context of abstract deformations of compact Kaehler manifolds and submanifolds.

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1. Introduction

This paper is an attempt to simplify, clarify, and extend some results about the interaction between deformation theory of Kähler manifolds and submanifolds and cohomology of the ambient manifold. It was occasioned by a review of the basics of Kuranishi theory and by the author's desire to reconstruct the results of preprints [R1–R4] in the setting of classical theory (see, for example, §3 of [GM]).

The basic idea in this paper is always the same, namely:

Let M_0 be a compact Kähler manifold. Since the variations of Hodge structure $H^*(M_0)$ over Artinian schemes always extend, therefore the subspaces of

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$H^q\left(\Omega^p_{M_0}\right)$ associated to a particular geometric deformation problem must pair to zero with the obstruction group for that problem.

The "therefore" in the above assertion is not obvious and is based on the equivalence of two pieces of data:

(1) The Gauss–Manin connection comparing the (trivial) topological deformation to the (non-trivial) deformation of Hodge structure.
(2) The Kuranishi data associated to a topological trivialization of a deformation of complex structure.

Thus the main thrust of this paper is establishing the precise link between the Kuranishi data and the Gauss–Manin connection.

The main use of the fact that obstructions pair to zero with certain cohomology classes is to reduce the size of the obstruction space for particular deformation problems. There are two main cases, many aspects of which have already been treated by other authors (e.g. [B,BF,FM,Ka,R1–R4,Ti,To]). So the purpose here is to clarify and refine what these authors and others have pointed out, namely that certain natural pairings between ambient Hodge classes and obstructions measure nothing more than the obstructions to deforming Hodge structures and therefore must vanish by Deligne [D]. This vanishing then gives useful limitations on the size of the obstruction space in question.

Case 1: The obstructions Obs to deforming a compact Kähler manifold M_0 annihilate the cohomology of M_0, that is,

$$\mathrm{Obs}\otimes H^{p,q}\left(M_0\right)$$

lies in the nullspace of the natural pairing

$$H^2\left(T_{M_0}\right)\otimes H^{p,q}\left(M_0\right)\to H^{p-1,q+2}\left(M_0\right).$$

(Recently, Manetti has given an elegant proof of this fact from the point of view of differential graded Lie algebras. See [M].)

Case 2: Given a deformation M/Δ of a compact Kähler manifold M_0, and given a compact submanifold Y_0 such that the sub-Hodge-structure

$$K_0^r=\sum K_0^{p,q}=\ker\left(H^r\left(M_0\right)\to H^r\left(Y_0\right)\right)$$

deforms over Δ for some r, then obstructions Obs to deforming Y_0 over Δ annihilate the primitive rth cohomology of M_0, that is,

$$\mathrm{Obs}\otimes K_0^{p,q}$$