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Advances in Mathematics 190 (2005) 161–195
ADVANCES IN Mathematics
http://www.elsevier.com/locate/aim

Central invariants and Frobenius–Schur indicators for semisimple quasi-Hopf algebras

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Received 7 May 2003; accepted 17 December 2003

Communicated by P. Etingof

Abstract

In this paper, we obtain a canonical central element ν_H for each semi-simple quasi-Hopf algebra H over any field k and prove that ν_H is invariant under gauge transformations. We show that if k is algebraically closed of characteristic zero then for any irreducible representation of H which affords the character χ, $\chi(\nu_H)$ takes only the values 0, 1 or -1, moreover if H is a Hopf algebra or a twisted quantum double of a finite group then $\chi(\nu_H)$ is the corresponding Frobenius–Schur indicator. We also prove an analog of a theorem of Larson–Radford for split semi-simple quasi-Hopf algebras over any field k. Using this result, we establish the relationship between the antipode S, the values of $\chi(\nu_H)$, and certain associated bilinear forms when the underlying field k is algebraically closed of characteristic zero.

Keywords: Frobenius–Schur indicators; Gauge equivalence; Tensor categories; Quasi-Hopf algebras

1. Introduction

In the paper [20], Linchenko and Montgomery introduced and studied Frobenius–Schur indicators for irreducible representations of a semi-simple Hopf algebra H

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[1] Research supported in part by NSF Grant DMS 0245225.

0001-8708/$ - see front matter
doi:10.1016/j.aim.2003.12.004

over an algebraically closed field of characteristic $p \neq 2$. If Λ is the *unique* normalized left integral of H, i.e. $\epsilon(\Lambda) = 1$, set

$$v = v_H = \sum_{(\Lambda)} \Lambda_1 \Lambda_2. \tag{1.1}$$

Here we have used Sweedler notation $\Delta(\Lambda) = \sum_{(\Lambda)} \Lambda_1 \otimes \Lambda_2$, so that if m is multiplication in H then $v = m \circ \Delta(\Lambda)$. Then v is a central element of H and the Frobenius–Schur indicator v_χ of an irreducible H-module M with character χ is defined via

$$v_\chi = \chi(v). \tag{1.2}$$

In case H is a group algebra $k[G]$, $v = |G|^{-1} \sum_{g \in G} g^2$ and $v_\chi = |G|^{-1} \sum_{g \in G} \chi(g^2)$ reduces to the original definition of Frobenius and Schur (cf. [6] or [27], for example). Generalizing the famous result of Frobenius and Schur for group algebras, Linchenko and Montgomery show that for general semi-simple H, v_χ can take only the values 0, 1, or -1. Moreover $v_\chi \neq 0$ if, and only if, $M \cong M^*$, and in this case M admits a non-degenerate H-invariant bilinear form $\langle \cdot, \cdot \rangle$ satisfying

$$\langle u, v \rangle = v_\chi \langle v, u \rangle \tag{1.3}$$

for $u, v \in M$. Recall that $\langle \cdot, \cdot \rangle$ is H-invariant if

$$\sum_{(h)} \langle h_1 u, h_2 v \rangle = \varepsilon(h) \langle u, v \rangle \tag{1.4}$$

for $h \in H$ and $u, v \in M$.

In a recent paper [16] the authors showed how one may effectively compute Frobenius–Schur indicators for a certain class of Hopf algebras. Their work applies, in particular, to the case of the *quantum double* $D(G)$ of a finite group G, and it was shown [16] how the indicators for irreducible modules over $D(G)$ may be given in terms of purely group-theoretic invariants associated to G and its subgroups. The algebra $D(G)$ is of interest in orbifold conformal field theory [23], indeed in this context there is a more general object, the *twisted quantum double* $D^\omega(G)$, that arises naturally [7]. (Here, $\omega \in Z^3(G, \mathbb{C}^\times)$ is a normalized 3-cocycle about which we shall have more to say below.) The present work originated with a natural problem: understand Frobenius–Schur indicators for twisted quantum doubles.

$D^\omega(G)$ is a semi-simple *quasi-Hopf* algebra (over $\mathbb{C}$, say), but is generally not a Hopf algebra. One of the difficulties this imposes is that the antipode S is not necessarily involutive (something that is always true for semi-simple Hopf algebras by a theorem of Larson and Radford [18]), whereas having $S^2 = id$ is fundamental for the Linchenko–Montgomery approach and therefore for the calculations in [16]. If it happens that $S^2 = id$ then Theorem 4.4 of [16] can be used to obtain indicators