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Generalized Fresnel integrals

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Abstract

A general class of (finite dimensional) oscillatory integrals with polynomially growing phase functions is studied. A representation formula of the Parseval type is proven as well as a formula giving the integrals in terms of analytically continued absolutely convergent integrals. Their asymptotic expansion for "strong oscillations" is given. The expansion is in powers of $\hbar^{1/2M}$, where $\hbar$ is a small parameters and $2M$ is the order of growth of the phase function. Additional assumptions on the integrands are found which are sufficient to yield convergent, resp. Borel summable, expansions.

Résumé

On étudie une classe générale d'intégrales oscillatoires en dimension finie avec une fonction de phase à croissance polynomiale. Une formule de représentation du type Parseval est démontrée, ainsi qu'une formule donnant les intégrales au moyen de la continuation analytique d'intégrales absolument convergentes. On donne les développements asymptotiques de ces intégrales dans le cas d' "oscillations rapides". Ces développements sont en puissance de $\hbar^{1/2M}$, où $\hbar$ est un petit paramètre et $2M$ est l'ordre de croissance de la fonction de phase. Sous des conditions additionelles sur les integrands on obtient la convergence, resp. la sommabilité au sens de Borel, des développements asymptotiques.

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1. Introduction

The study of finite dimensional oscillatory integrals of the form

$$\int_{\mathbb{R}^N} e^{\frac{i}{\hbar}\Phi(x)} f(x)\,dx, \tag{1}$$

where $\hbar$ is a non vanishing real parameter, Φ and f suitable real-valued smooth functions, is already a classical topic, largely developed in connections with various problems in mathematics and physics. Well known examples of simple integrals of the above form are the Fresnel integrals of the theory of wave diffraction and Airy's integrals of the theory of rainbow. The theory of Fourier integral operators [25,26,34] also grew out of the investigation of oscillatory integrals. It allows the study of existence and regularity of a large class of elliptic and pseudoelliptic operators and provides constructive tools for the solutions of the corresponding equations. In particular one is interested in discussing the asymptotic behavior of the above integrals when the parameter $\hbar$ goes to 0. The method of stationary phase provides a tool for such investigations and has many applications, such as the study of the classical limit of quantum mechanics (see [2,3,8,22,33,44]). In the general case of degenerate critical points of the phase function Φ, the theory of unfoldings of singularities is applied, see [13,20].

The extensions of the definition of oscillatory integrals to an infinite dimensional Hilbert space $\mathcal{H}$ and the implementation of a corresponding infinite-dimensional version of the stationary phase method has a particular interest in connection with the rigorous mathematical definition of the "Feynman path integrals". Several methods has been discussed in literature, for instance by means of analytic continuation of Wiener integrals [16,17, 29,30,32,35,36,42,43], or by "infinite dimensional distributions" in the framework of the Hida calculus [19,24], via "complex Poisson measures" [1,34], via a "Laplace transform method" [5,31], or via a "Fourier transform approach", see [3,4,6–8,21,27,28]. The latter method is particularly interesting as it is the only one by which a development of an infinite dimensional stationary phase method has been performed. The phase functions that can be handled by this method are of the form "quadratic plus bounded perturbation", that is $\Phi(x) = \langle x, Tx\rangle + V(x)$, where T is a self-adjoint operator and V is the Fourier transform of a complex bounded variation measure on $\mathcal{H}$ [3,8,40,41].

We also mention that the problem of definition and study of integrals of the form (1) but with $\hbar \in \mathbb{C}$, $\mathrm{Im}(\hbar) < 0$ and Φ lower bounded has also been discussed. The convergence of the integral in this case is a simple matter, so the analysis has concentrated on a "perturbation theoretical" computation of the integral, like in [14,15], resp. on a Laplace method for handling the $\hbar \to 0$ asymptotics, see, e.g. [3,9,11,12,38] (the latter method has some