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Stochastic control and compatible subsets of constraints ☆

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Abstract

Given a stochastic differential control system and a closed set K in $\mathbb{R}^n$, we study the that, with probability one, the associated solution of the control system remains for ever in the set K. This set is called the *viability kernel of* K. If N is equal to the whole set K, K is said to be viable. We prove that, in the general case, the viability kernel itself is viable and we characterize it through some partial differential equations. We prove that, under suitable assumptions, also the boundary of N is viable. As an application, we give a new characterization of the value function of some optimal control problem.

Résumé

Etant donné un système de contrôle stochastique et un ensemble fermé K dans $\mathbb{R}^n$, nous étudions l'ensemble $N \subset K$ des points x, pour lesquels il existe un contrôle v tel que, avec probabilité 1, la trajectoire de la solution associée au système de contrôle reste pour toujours dans K. On appelle cet ensemble le noyau de viabilité de K. Lorsque $N = K$, on dit que K est viable. Nous montrons ici que, dans le cas général, le noyau de viabilité est viable et le caractérisons à l'aide d'une équation aux dérivées partielles. Nous montrons que, sous de bonnes hypothèses, le bord de N est ̧galement viable. Finalement, les résultats obtenus nous permettent de caractériser la fonction valeur d'un problème particulier de contrôle optimal.

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1. Introduction

Given a d-dimensional Brownian motion W on a probability space $(\Omega, \mathcal{F}, P)$, we consider a stochastic differential control system:

$$\begin{cases} dX^{x,v(\cdot)}(t) = b(X^{x,v(\cdot)}(t), v(t))\,dt + \sigma(X^{x,v(\cdot)}(t), v(t))\,dW(t), & t \in [0,\infty), \\ X^{x,v(\cdot)}(0) = x \in \mathbb{R}^n. \end{cases} \tag{1}$$

Let K a closed set in $\mathbb{R}^n$. We say that the constraint K is compatible with the control system (1) – or following terminology of [2], we say that K is *viable* for (1) – if and only if for any $x \in K$, there exist a control v such that the corresponding solution to (1) satisfies

$$P\text{-almost surely}, \quad \forall t \geqslant 0, \quad X^{x,v(\cdot)}(t) \in K. \tag{2}$$

In general, the set K is not viable for (1). It is then natural to interest oneself on the set of initial conditions $x \in K$, for which it is possible to find a control $v(\cdot)$, such that (2) holds. We call this set the viability kernel of K and we denote it by $\mathrm{Viab}_{(b;\sigma)}(K)$. The main aim of the present paper is to study it.

When K is *viable* for the system (1), we have obviously $K = \mathrm{Viab}_{(b;\sigma)}(K)$. The property of viability of K for the system (1) has been extensively studied. We refer the reader to the monography [1] for the deterministic case. For the stochastic case, several characterizations have been obtained: through stochastic tangent cones in [2,3,13], through viscosity solutions of partial differential equations in [6,7]. We mention also [8] for time-depending constraints and [9] for viability for backward stochastic differential equations. This property has been also investigated in slightly different contexts in [4,5,16–20].

The viable kernel plays an important role in deterministic control, for example to study solutions of first order Hamilton Jacobi equation (see [10] for references).

The present paper is devoted to the stochastic case. Our first result gives an equivalent definition of the viability kernel of K: it is the largest closed subset of K which is viable.

We investigate fine properties of the boundary of the viability kernel, in particular that the boundary of the kernel itself is viable. This is a generalization of similar properties obtained for deterministic control in [21,22].

We also investigate an optimal stochastic control of the form

$$\inf_{v(\cdot)} \big(\text{ess-sup}_\Omega\, g\big(X^{x,v(\cdot)}(T)\big)\big). \tag{3}$$

We prove that the epigraph of the associated value function is the viability kernel of a suitably extended stochastic control system. As a consequence, we give some new characterizations of this value function and several other consequences.

The paper is organized as follows: After some preliminaries in the first section, we devote the second section to prove that the viability kernel is viable and to obtain useful