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Analysis of restrictions of unitary representations of a nilpotent Lie group

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Abstract

Let G be a connected simply connected nilpotent Lie group, K an analytic subgroup of G and π an irreducible unitary representation of G. Let $D_\pi(G)^K$ be the algebra of differential operators keeping invariant the space of C^∞ vectors of π and commuting with the action of K on that space. In this paper, we assume that the restriction of π to K has finite multiplicities and we show that $D_\pi(G)^K$ is isomorphic to a subalgebra of the field of rational K-invariant functions on the co-adjoint orbit $\Omega(\pi)$ associated to π, and for some particular cases, that $D_\pi(G)^K$ is even isomorphic to the algebra of polynomial K-invariant functions on $\Omega(\pi)$. We prove also the Frobenius reciprocity for some restricted classes of nilpotent Lie groups, especially in the cases where K is normal or abelian.

Résumé

Soit G un groupe de Lie nilpotent connexe et simplement connexe, K un sous-groupe analytique de G et π une représentation unitaire et irréductible G. Soit $D_\pi(G)^K$ l'algèbre des opérateurs différentiels qui laissent invariant l'espace des vecteurs C^∞ de π et qui commutent avec l'action de K sur cet espace. Nous prouvons dans ce papier que sous l'hypothèse que la restriction de π à K est à multiplicités finies, l'algèbre $D_\pi(G)^K$ est isomorphe à une sous-algèbre du corps des fonctions rationnelles K-invariantes sur l'orbite co-adjointe $\Omega(\pi)$ associée à π, et dans certains cas particuliers

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que $D_\pi(G)^K$ est même isomorphe à l'algèbre des fonctions polynomiales K-invariantes sur $\Omega(\pi)$. Nous prouvons aussi la réciprocité de Frobenius pour quelques classes de groupes de Lie nilpotents, particulièrement les cas où K est normal ou abélien.

MSC: 22E27

Keywords: Nilpotent Lie group; Irreducible representation; Differential operator; Restricted representation

1. Introduction and notations

1.1. It is well known that there exists a strong parallelism between inductions and restrictions of representations of locally compact groups. Monomial representations of nilpotent Lie groups have been analyzed in detail: the canonical central disintegration in [4,9,16,27,28], Plancherel formula in [5,6,14,17,31], the associated algebra of invariant differential operators in [2,11,19–22,24] and the Frobenius reciprocity in [6,23,31].

Concerning the restriction, similar investigations have begun, but much less has been done so far: the canonical central disintegration has been studied in [10,18] and the associated algebra of invariant differential operators in [2,3]. In this paper we continue the analysis of the restriction by looking at Frobenius vectors and the Frobenius reciprocity.

1.2. Let $G = \exp(\mathfrak{g})$ be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. We denote by $\widehat{G}$ the unitary dual of G, i.e. the set of all equivalence classes of irreducible unitary representations of G. We shall sometimes identify the equivalence class $[\pi]$ with its representative π and we denote the equivalence relation between two representations π_1 and π_2 by $\pi_1 \simeq \pi_2$ or even by $\pi_1 = \pi_2$.

1.3. Let $\mathfrak{g}^*$ be the dual vector space of $\mathfrak{g}$. By Kirillov's orbit theory, $\widehat{G}$ can be realized as the space of coadjoint orbits $\mathfrak{g}^*/G$ by means of Kirillov's mapping $\Theta = \Theta_G : \mathfrak{g}^*/G \to \widehat{G}$ (cf. [7,26]). We designate by the same notation Θ its pull-back $\mathfrak{g}^* \to \widehat{G}$ too. Let us write $\Omega(\pi) = \Omega_G(\pi)$ for the Kirillov-orbit $\Theta^{-1}(\pi)$ of π and also π_l, $l \in \mathfrak{g}^*$, for the irreducible representation $\Theta_G(G \cdot l)$.

1.4. Let π be an irreducible unitary representation of G. We restrict π to an analytic subgroup $K = \exp(\mathfrak{k})$ of G and we denote by $\pi_{|K}$ this restriction.

1.5. Let us recall the canonical central disintegration of $\pi_{|K}$ (cf. [10,18]). Let μ_π be a finite measure on $\mathfrak{g}^*$, which is equivalent to the G-invariant measure on the coadjoint orbit $\Omega(\pi)$. We consider the image $\mu = (\Theta_K \circ p)_*(\mu_\pi)$ of μ_π under the mapping $\Theta_K \circ p : \mathfrak{g}^* \to \widehat{K}$, where $p : \mathfrak{g}^* \to \mathfrak{k}^*$ stands for the canonical projection. For $\sigma \in \widehat{K}$, let $n_\pi(\sigma)$ be the number of K-orbits contained in $\Gamma(\pi, \sigma) = \Omega_G(\pi) \cap p^{-1}(\Omega_K(\sigma))$. Then

$$\pi_{|K} \simeq \int_{\widehat{K}}^{\oplus} n_\pi(\sigma)\sigma \, d\mu(\sigma). \tag{1.5.1}$$