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C. R. Acad. Sci. Paris, Ser. I 341 (2005) 29–34

COMPTES RENDUS MATHEMATIQUE

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Géométrie différentielle

Quasi-morphisme de Calabi sur les surfaces de genre supérieur

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Reçu le 24 mai 2005 ; accepté le 28 mai 2005

Présenté par Étienne Ghys

Résumé

Nous construisons un quasi-morphisme homogène $\mathfrak{Cal}_S$ sur le groupe des difféomorphismes hamiltoniens d'une surface (fermée, connexe, orientée) de genre supérieur ou égal à 2, ayant la propriété suivante. Si U est un ouvert connexe de S difféomorphe à un disque ou à un anneau, la restriction de $\mathfrak{Cal}_S$ au sous-groupe formé des difféomorphismes qui sont le temps 1 d'une isotopie hamiltonienne dans U, est égale au morphisme de Calabi. ***Pour citer cet article : P. Py, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Abstract

Calabi quasi-morphism on higher genus surfaces. We construct a homogeneous quasi-morphism $\mathfrak{Cal}_S$ on the group of Hamiltonian diffeomorphisms of a (closed, connected, oriented) surface S of genus greater or equal to 2, with the following property. For each connected open set U in S diffeomorphic to a disk or to an annulus, the restriction of $\mathfrak{Cal}_S$ to the subgroup of diffeomorphisms which are the time 1 map of a Hamiltonian isotopy in U, equals Calabi's homomorphism. ***To cite this article: P. Py, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Abridged English version

A *quasi-morphism* on a group Γ is a function $\phi:\Gamma\to\mathbb{R}$, such that the quantity

$$\sup_{x,y\in\Gamma}\left|\phi(xy)-\phi(x)-\phi(y)\right|$$

is finite. The quasi-morphism ϕ is *homogeneous*, if it satisfies moreover $\phi(x^n)=n\phi(x)$ for $x\in\Gamma$ and $n\in\mathbb{Z}$ (see [3] for an introduction to this subject).

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2005.06.002

Given a closed connected symplectic manifold (V,ω), the group $\mathrm{Ham}(V,\omega)$ of Hamiltonian diffeomorphisms of V is a simple group, according to a theorem of Banyaga [2]. Hence there is no non-trivial homomorphism from $\mathrm{Ham}(V,\omega)$ to $\mathbb{R}$. If (V,ω) is a connected open symplectic manifold, and if ω is exact on V, Calabi defined in [5] a homomorphism

$$\mathfrak{Cal}_V : \mathrm{Ham}(V,\omega) \to \mathbb{R},$$

the kernel of which is a simple group, according to another theorem of Banyaga [2]. If λ is a primitive of ω and is (f_t) is a Hamiltonian isotopy generated by the vector field Z_t, one has:

$$\mathfrak{Cal}_V(f_1) = \int_V \int_0^1 \lambda(Z_t)\,\mathrm{d}t\,\omega.$$

Suppose now that V is closed. To each connected open set $U \subset V$, one can associate the subgroup Γ_U of $\mathrm{Ham}(V,\omega)$ consisting of all the diffeomorphisms which are the time 1 map of a Hamiltonian isotopy in U. If ω is exact on U, we have Calabi's homomorphism $\mathfrak{Cal}_U : \Gamma_U \to \mathbb{R}$. We shall denote by $\mathcal{D}$ the family of connected open sets U such that ω is exact on U, and there exists $f \in \mathrm{Ham}(V,\omega)$ with $f(U) \cap \overline{U} = \emptyset$. In [7], Entov and Polterovich raise the following problem:

Can we construct a homogeneous quasi-morphism $\phi : \mathrm{Ham}(V,\omega) \to \mathbb{R}$, whose restrictions to the subgroups $(\Gamma_U)_{U\in\mathcal{D}}$ equal Calabi's homomorphisms $(\mathfrak{Cal}_U)_{U\in\mathcal{D}}$?

More generally, can we construct such a quasi-morphism on the universal cover $\widetilde{\mathrm{Ham}}(V,\omega)$ of the group of Hamiltonian diffeomorphisms? They give a positive answer to this question for a certain class of symplectic manifolds, including complex projective spaces, in particular the 2-sphere, and using hard tools from symplectic topology. Using methods in the spirit of the constructions in [8], we prove:

Theorem 0.1. *If (S,ω) is a closed oriented surface of genus g greater or equal to 2, endowed with a symplectic form, there exists a homogeneous quasi-morphism $\mathfrak{Cal}_S : \mathrm{Ham}(S,\omega) \to \mathbb{R}$, whose restriction to the subgroup Γ_U equals Calabi's homomorphism, as soon as U is diffeomorphic to a disk or to an annulus. The quasi-morphism $\mathfrak{Cal}_S$ is invariant under conjugacy by symplectic diffeomorphisms.*

Note that many quasi-morphisms on the group $\mathrm{Ham}(S,\omega)$ (for every closed oriented surface) were constructed in [8], but their restrictions to the subgroups Γ_U are not homomorphisms. The nature of our quasi-morphism is certainly quite different from that of Entov and Polterovich, constructed in [7] (see also [4]). Indeed, the condition we require about the open set U is different from the two conditions defining the family $\mathcal{D}$. In dimension 2, the symplectic form is always exact on an open set different from S, but some disks do not satisfy the second one.

The following theorem is motivated by theorem 5.2 in [7], where a similar result is proved on the 2-sphere. We consider a Morse function $F : S \to \mathbb{R}$. We denote by $x_1,\dots,x_l$ its critical points, $\lambda_j = F(x_j)$ its critical values, and we suppose $\lambda_1 < \cdots < \lambda_l$. Set $\mathcal{F} = \{H:\ S \to \mathbb{R},\ \omega(X_H, X_F) = 0\}$, the space of functions commuting with F (X_H is the Hamiltonian vector field associated with H). The set:

$$\Gamma = \{\varphi_H^1,\ H \in \mathcal{F}\},$$

of time 1 maps of Hamiltonian flows generated by elements in $\mathcal{F}$, is an Abelian subgroup of $\mathrm{Ham}(S,\omega)$. The restriction of the quasi-morphism $\mathfrak{Cal}_S$ to Γ is a homomorphism. We associate a finite graph $\mathcal{G}$ to the Morse function F in the following way. The connected components of level sets $F^{-1}(t)$ are of three kinds:

(i) critical points of F of index 0 or 2;
(ii) simple closed curves;
(iii) immersed closed curves with one self-intersection (corresponding to a critical point of index one of F).