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Analyse numérique/Problèmes mathématiques de la mécanique

Inégalités d'entropie pour un schéma de relaxation

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Résumé

Nous nous intéressons aux inégalités discrètes d'entropie pour une classe de schémas de relaxation. Après une brève description de la méthode, nous proposons une démonstration *directe* pour établir les inégalités discrètes d'entropie. Ces inégalités sont, en fait, la conséquence d'un principe de minimisation de l'entropie satisfait par le modèle de relaxation considéré. Ces résultats sont ensuite étendus au modèle aux 10 moments. ***Pour citer cet article : C. Berthon, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Abstract

Entropy inequalities for a relaxation scheme. This work is devoted to the discrete entropy inequalities when considering relaxation schemes. After describing the numerical method, we propose a *direct* proof to establish the discrete entropy inequalities. In fact, we show that the considered relaxation model satisfies a minimum principle on the entropy. This principle implies the expected inequalities. The work is concluded when applying the above results to the 10 moment model. ***To cite this article: C. Berthon, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Abridged English version

We consider the numerical approximation of the weak solutions of the usual 3×3 Euler equations:

$$\partial_t \mathbf{v} + \partial_x \mathbf{f}(\mathbf{v}) = 0, \quad \mathbf{v} = {}^t(\rho, \rho u, \rho E), \quad \mathbf{f}(\mathbf{v}) = {}^t\big(\rho u, \rho u^2 + p, (\rho E + p)u\big), \tag{1}$$

with $\rho E = \rho u^2/2 + \rho e$ where $e > 0$ denotes the internal energy. The state law $p := p(\rho, e) > 0$ is assumed to be associated with a convex entropy $\rho s(\rho, e)$ which satisfies the following inequality:

$$\partial_t \rho s + \partial_x \rho s u \leqslant 0. \tag{2}$$

The map $e \to \mathrm{s}(\rho, e)$ will be decreasing. In the present work, we will not consider the vacuum problem.

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To approximate the weak solutions of (1) and (2), we propose to consider a suitable extended first order system with singular perturbations. The aim of this system is to restore the initial system in the regime of an infinite relaxation parameter. Several choices can be considered [1,8,11] to define the first order system with singular perturbations, denoted *relaxation system* in the sequel. Motivated by [5,10], we consider a relaxation system such that most of the nonlinearities of the Euler equations are kept for the sake of accuracy. We propose to relevantly modify the pressure law [11]:

$$\begin{aligned}
&\partial_t \mathbf{w}^\lambda + \partial_x \mathbf{g}(\mathbf{w}^\lambda) = \lambda \mathbf{R}(\mathbf{w}^\lambda),\\
&\mathbf{w}^\lambda = {}^t(\rho^\lambda, \rho^\lambda u^\lambda, \rho^\lambda E^\lambda, \rho^\lambda \pi^\lambda), \qquad \mathbf{R}(\mathbf{w}^\lambda) = {}^t(0,0,0,p(\rho^\lambda, e^\lambda) - \pi^\lambda),\\
&\mathbf{g}(\mathbf{w}^\lambda) = {}^t\big(\rho^\lambda u^\lambda, \rho^\lambda (u^\lambda)^2 + \pi^\lambda, (\rho^\lambda E^\lambda + \pi^\lambda) u^\lambda, \rho^\lambda \pi^\lambda u^\lambda + a^2 u^\lambda\big).
\end{aligned} \tag{3}$$

For the sake of clarity in the notations, we will omit the exponent λ. The relaxation parameter $a > 0$ must satisfy the sub-characteristic Whitham condition [13]:

$$a^2 > \rho^2 \partial_\rho p(\rho, e) + p(\rho, e) \partial_e p(\rho, e). \tag{4}$$

This condition is actually needed to prevent the relaxation approximation procedure from instabilities as λ goes to infinity. Let us note from now on that the system $(11)_{\lambda=0}$ is nonlinear hyperbolic. All the fields of the system are linearly degenerated. The Jacobian matrix of the flux function $\mathbf{g}$ admits as eigenvalues: u, $u \pm a/\rho$. As a consequence, the solution of the Riemann problem is made of constant states separated by contact discontinuities. In addition, as soon as the relaxation parameter a is assumed to be large enough, the density and the internal energy of the Riemann solution remain positive. In the sequel, this condition to ensure the positiveness of the density and the internal energy, will be systematically assumed: a is large enough to satisfy,

$$\frac{1}{\rho_L} + \frac{u_\star - u_L}{a} > 0, \quad \frac{1}{\rho_R} + \frac{u_R - u_\star}{a} > 0, \quad e_L + \frac{\pi_\star^2 - \pi_L^2}{2a^2} > 0, \quad e_R + \frac{\pi_\star^2 - \pi_R^2}{2a^2} > 0, \tag{5}$$

with $u_\star = \frac{u_L + u_R}{2} + \frac{\pi_L - \pi_R}{2a}$ and $\pi_\star = \frac{\pi_L + \pi_R}{2} + \frac{a}{2}(u_L - u_R)$, where $\mathbf{w}_L$ and $\mathbf{w}_R$ denote the left and right states of the initial Riemann problem.

Now, based on the relaxation system (3), we propose a relaxation scheme to approximate the solutions of (1) and (2) and we recall the aim of the method [1,3,4]. Let an approximation of the equilibrium solution at the date t^n: $\mathbf{v}^n(x) = \mathbf{v}_i^n$ if $x \in (x_{i-1/2}, x_{i+1/2})$ be given. We propose to evolve this approximation at the next time level into two steps.

Evolution: $t^n \to t^{n+1,-}$. With $0 < t < \Delta t$, we solve the Cauchy problem $(11)_{\lambda=0}$ with the initial data given by $\mathbf{w}^n(x) = \mathbf{w}_i^n$ if $x \in (x_{i-1/2}, x_{i+1/2})$ where the approximation at the equilibrium is ensured by $\mathbf{w}_i^n = {}^t(\rho_i^n, (\rho u)_i^n, (\rho E)_i^n, (\rho\pi)_i^n)$ with $\pi_i^n = p(\rho_i^n, e_i^n)$. As soon as the following CFL condition is satisfied:

$$\frac{\Delta t}{\Delta x} \max_{i \in \mathbb{Z}} \big(|u|, |u \pm a/\rho|\big) \leqslant \frac{1}{2}, \tag{6}$$

the solution $\mathbf{w}^h(x,t)$ is made of the juxtaposition, with no interaction, of the Riemann problem solutions set at each interface $x_{i+1/2}$. Then, we define $\mathbf{w}_i^{n+1,-} = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \mathbf{w}^h(x, \Delta t)\, dx$. In this first step of the method, let us note that the parameter a can be locally evaluated on each Riemann problem. This evaluation must satisfy the Whitham condition (4) and the positiveness condition (5). As a consequence, we immediately obtain $\rho_i^{n+1,-} > 0$.

Relaxation: $t^{n+1,-} \to t^{n+1}$. A projection of the vector state $\mathbf{w}_i^{n+1,-}$ is done on the equilibrium manifold $\{\pi = p(\rho, e)\}$. Put in other words, we solve $\partial_t \mathbf{w} = \lambda \mathbf{R}(\mathbf{w})$ with $\mathbf{w}_i^{n+1,-}$ as the initial data and we consider the solution as λ tends to ∞. We obtain $\mathbf{v}_i^{n+1} = {}^t(\rho, \rho u, \rho E)_i^{n+1,-}$ and $\pi_i^{n+1} = p(\rho_i^{n+1}, e_i^{n+1})$.

To conclude the presentation of the relaxation scheme, we recall that this scheme typically enters the framework of the usual conservative finite volume method: