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Analyse numérique

Approximation numérique d'un problème de membrane non linéaire

Nabil Kerdid [a], Hervé Le Dret [b], Abdelkader Saïdi [c]

[a] *College of Computer Science and Information Systems, Imam University, Riyadh, Saudi Arabia*
[b] *Laboratoire Jacques-Louis Lions, université Pierre et Marie Curie, boîte courrier 187, 75252 Paris cedex 05, France*
[c] *Institut de recherche mathématique avancée, université Louis Pasteur, 7, rue René Descartes, 67084 Strasbourg, France*

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Présenté par Philippe G. Ciarlet

Résumé

On étudie numériquement les déformations d'une membrane élastique non linéaire. On considère le modèle de membrane obtenu par Le Dret et Raoult par la méthode de Γ-convergence. Les déformations de la membrane minimisent une énergie non quadratique. On effectue une approximation du modèle par éléments finis conformes et on utilise un algorithme de gradient conjugué non linéaire pour minimiser l'énergie discrétisée. ***Pour citer cet article : N. Kerdid et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Abstract

Numerical approximation for a nonlinear membrane problem. We study numerically the deformations of a nonlinearly elastic membrane. We consider the nonlinear membrane model obtained by Le Dret and Raoult using Γ-convergence. In this model, membrane deformations minimize a highly nonquadratic energy. We consider a conforming finite element approximation of the problem and use a nonlinear conjugate gradient algorithm to minimize the discrete energy. ***To cite this article: N. Kerdid et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Adresses e-mail : nkerdid@imamu.edu.sa, kerdid@ann.jussieu.fr (N. Kerdid), ledret@ccr.jussieu.fr (H. Le Dret), saidi@math.u-strasbg.fr (A. Saïdi).

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Abridged English version

The continuous problem

We consider a nonlinearly elastic membrane with midsurface ω, where ω is an open, bounded subset of $\mathbb{R}^2$ with Lipschitz boundary.

We assume that the membrane is made of a bulk material that is homogeneous and hyperelastic, with stored energy function $W : M_3 \to \mathbb{R}$ where M_3 is the space of real 3×3 matrices. The function W is assumed to be continuous and coercive. Let $M_{3,2}$ be the space of real 3×2 matrices. If z_α, $\alpha = 1, 2$, are two vectors in $\mathbb{R}^3$, we note $(z_1|z_2)$ the 3×2 matrix whose columns are z_α. For all $F = (z_1|z_2) \in M_{3,2}$ and $z \in \mathbb{R}^3$, we note $(F|z)$ the 3×3 matrix whose first two columns are z_α and third column is z.

We define a function W_0: $M_{3,2} \to \mathbb{R}$ by $W_0((z_1|z_2)) = \inf_{z\in\mathbb{R}^3} W((z_1|z_2|z))$, see [3]. The function W_0 is continuous and coercive. We denote by QW_0 its quasiconvex envelope. We introduce the space of admissible membrane displacements,

$$\Phi_M = \{\psi \in W^{1,p}(\omega; \mathbb{R}^3);\ \psi(x_1, x_2) = (x_1, x_2, 0)^{\mathrm{T}} \text{ on } \partial\omega\}. \tag{1}$$

We define the nonlinear membrane energy for all $\psi \in \Phi_M$ by

$$J(\psi) = 2\int_\omega QW_0(\nabla\psi)\,\mathrm{d}x_1\,\mathrm{d}x_2 - \int_\omega f\cdot\psi\,\mathrm{d}x_1\,\mathrm{d}x_2, \tag{2}$$

where f denotes a given force resultant density.

In [3], Le Dret and Raoult computed the Γ-limit of the three-dimensional energy when the thickness of the membrane goes to zero, thereby showing that the deformations that minimize the energy of the three-dimensional problem converge, in an appropriate sense, toward solutions of the following two-dimensional minimization problem: Find $\phi \in \Phi_M$ such that

$$J(\phi) = \inf_{\psi\in\Phi_M} J(\psi). \tag{3}$$

The energy density QW_0 is quasiconvex and existence of a solution to problem (3) is guaranteed under reasonable technical assumptions, see [3].

In the case of a Saint Venant–Kirchhoff material, the limit membrane stored energy function can be computed explicitly (see [3]) as function of the right singular values $v_1 \leqslant v_2$ of $F \in M_{3,2}$, It turns out to be equal to the convex envelope of the function W_0 in this particular case. The goal of this Note is to present a numerical method to approximate the solutions of the minimization problem.

The discrete problem

Let τ_h be a regular affine family of triangulations covering the (polygonal) domain ω. We discretize the three Cartesian components of the deformation using P_1 finite elements. The discrete space of admissible displacements is given by

$$\Phi_M^h = \{\psi_h \in C^0(\omega; \mathbb{R}^3), \psi_h(x_1, x_2) = (x_1, x_2, 0)^{\mathrm{T}} \text{ on } \partial\omega, \psi_{h|_K} \in (P_1)^3;\ \forall K \in \tau_h\}. \tag{4}$$

Clearly, $\Phi_M^h \subset \Phi_M$, and the approximation is conforming.

To numerically minimize J over Φ_M^h, we use the nonlinear conjugate gradient method. We choose the Polak and Ribière [5] variant thereof, which gives good results.

We proceed as follows to compute ∇J. Let $(\lambda_j)_{j=1,\dots,N_h}$ be the components of ψ_h in the shape function basis $(\Theta_j)_{j=1,\dots,N_h}$. We have:

$$\frac{\partial J}{\partial\lambda_j}(\psi_h) = \int_\omega \frac{\partial(QW_0)}{\partial F}(\nabla\psi_h) : \nabla\Theta_j\,\mathrm{d}x_1\,\mathrm{d}x_2 - \int_\omega f\cdot\Theta_j\,\mathrm{d}x_1\,\mathrm{d}x_2. \tag{5}$$