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Two new properties of ideals of polynomials and applications ☆

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ABSTRACT

In this paper we introduce two properties for ideals of polynomials between Banach spaces and show how useful they are to deal with several a priori different problems. By investigating these properties we obtain, among other results, new polynomial characterizations of $\mathcal{L}_\infty$ spaces and characterizations of Banach spaces whose duals are isomorphic to $\ell_1(\Gamma)$.

1. INTRODUCTION, NOTATIONS AND BACKGROUND

The concept of operator ideals goes back to Grothendieck [10] and its natural extension to polynomials and multilinear mappings is credited to Pietsch [16]. For references on operator ideals we refer to the book by Pietsch [15]. In this paper we identify two properties concerning ideals of polynomials between Banach spaces and by exploring these properties we give an unified treatment to some questions that (until now) are being investigated separately. Among other results, we prove new polynomial characterizations of $\mathcal{L}_\infty$-spaces and spaces whose duals are isomorphic to $\ell_1(\Gamma)$, extending results of Lewis and Stegall [11], Stegall [17], Stegall and Retherford [18] and Cilia, D'Anna and Gutiérrez [5,6].

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Throughout this paper $E, E_1, \ldots, E_n, F, G, G_1, \ldots, G_n, H$ will stand for (real or complex) Banach spaces, E' denotes the topological dual of E and B_E represents the closed unit ball of E. The symbol $\mathbb{K}$ will represent the field of all scalars and $\mathbb{N}$ denotes the set of all natural numbers. Given a natural number $n \geqslant 1$, the Banach space of all continuous n-linear mappings from $E_1 \times \cdots \times E_n$ into F endowed with the sup norm will be denoted by $\mathcal{L}(E_1, \ldots, E_n; F)$ and (if $n \geqslant 2$) the Banach space of all continuous n-homogeneous polynomials P from E into F with the sup norm is denoted by $\mathcal{P}(^nE; F)$. For the general theory of polynomials/multilinear mappings we refer to [8] and [14].

Ideals of multilinear functionals were introduced by Pietsch [16] and, as far as we know, ideals of polynomials appeared for the first time in Braunss [3].

Definition 1. An *ideal of multilinear mappings* $\mathcal{M}$ is a subclass of the class of all continuous multilinear mappings between Banach spaces such that for all n and $E_1, \ldots, E_n$, the components $\mathcal{M}(E_1, \ldots, E_n; F) = \mathcal{L}(E_1, \ldots, E_n; F) \cap \mathcal{M}$ satisfy:

(i) $\mathcal{M}(E_1, \ldots, E_n; F)$ is a linear subspace of $\mathcal{L}(E_1, \ldots, E_n; F)$ which contains the n-linear mappings of finite type.
(ii) If $A \in \mathcal{M}(E_1, \ldots, E_n; F)$, $u_j \in \mathcal{L}(G_j; E_j)$ for $j = 1, \ldots, n$ and $\varphi \in \mathcal{L}(F; H)$, then $\varphi \circ A \circ (u_1, \ldots, u_n) \in \mathcal{M}(G_1, \ldots, G_n; H)$.

When there exists $\|.\|_{\mathcal{M}} : \mathcal{M} \to [0, \infty[$ satisfying

(i′) $\|.\|_{\mathcal{M}}$ restricted to $\mathcal{M}(E_1, \ldots, E_n; F)$ is a norm (respectively, quasi-norm) for all $E_1, \ldots, E_n, F$ and all natural numbers n,
(ii′) $\|A : \mathbb{K}^n \to \mathbb{K};\ A(x_1, \ldots, x_n) = x_1 \cdots x_n\|_{\mathcal{M}} = 1$ for all n,
(iii′) if $A \in \mathcal{M}(E_1, \ldots, E_n; F)$, $u_j \in \mathcal{L}(G_j; E_j)$ for $j = 1, \ldots, n$ and $\varphi \in \mathcal{L}(F; H)$, then

$$\left\|\varphi \circ A \circ (u_1, \ldots, u_n)\right\|_{\mathcal{M}} \leqslant \|\varphi\| \|A\|_{\mathcal{M}} \|u_1\| \cdots \|u_n\|,$$

$\mathcal{M}$ is called a *normed* (respectively, *quasi-normed*) *ideal of multilinear mappings*.

Fixed an ideal of multilinear mappings $\mathcal{M}$ and $n \in \mathbb{N}$, the class $\mathcal{M}_n = \bigcup_{E_1, \ldots, E_n, F} \mathcal{M}(E_1, \ldots, E_n; F)$ is called an *ideal of n-linear mappings*.

An *ideal of (homogeneous) polynomials* $\mathcal{Q}$ is a subclass of the class of all continuous homogeneous polynomials between Banach spaces such that for all $n \in \mathbb{N}$ and all Banach spaces E, F, the components $\mathcal{Q}(^nE; F) = \mathcal{P}(^nE; F) \cap \mathcal{Q}$ satisfy:

(i) $\mathcal{Q}(^nE; F)$ is a linear subspace of $\mathcal{P}(^nE; F)$ which contains the polynomials of finite type.
(ii) If $P \in \mathcal{Q}(^nE; F)$, $\varphi_1 \in \mathcal{L}(G; E)$ and $\varphi_2 \in \mathcal{L}(F; H)$, then $\varphi_2 \circ P \circ \varphi_1 \in \mathcal{Q}(^nG; H)$.